Paper Code: Semester-II

ALGEBRA AND TRIGONOMETRY, ORDINARY DIFFERENTIAL EQUATIONS, VECTOR CALCULUS

Bachelor of Arts (B.A.)

Three Year Programme

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CHAPTER – I

SUCCESSIVE DIFFERENTIATION

Remarks

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1.1 INTRODUCTION

If y be a function of x i.e. f(x). Then $\frac{dy}{dx} = f'(x)$ is called 1^{st} derivative of y w.r.t. x and $\frac{d^2y}{dx^2} = f''(x)$ is called 2^{nd} derivative of y w.r.t. x and $\frac{d^3y}{dx^3} = f'''(x)$ is card 3^{rd} derivative of y w.r.t. x and so on. In general, the n^{th} differential co-efficients of y i.e. n^{th} derivative of y w.r.t. x is denoted by $\frac{d^ny}{dx^n}$.

The process of finding the differential co-efficients of a function is called successive differentiation.

1.2 OBJECTIVE

After reading this lesson, you must be able to understand

- n^{th} derivatives.
- n^{th} derivatives (a) By partial fraction (b) By Leibnitz theorem.
- n^{th}derivatives at x = 0.

Example 1 : (i) If y = cot x then find $\frac{d^3y}{dx^3}$.

(ii) If $y = x^3 \sin ax$. Find $\frac{d^2y}{dx^2}$

Solution : (i) Let y = cot x

$$\frac{dy}{dx} = -\operatorname{cosec}^2 x$$

$$\frac{d^2y}{dx^2} = -2\operatorname{cosec} x(-\operatorname{cosec} x \cot x)$$

$$= 2\operatorname{cosec}^2 x \cot x = 2(1 + \cot^2) \cot x$$

$$= 2(\cot x + \cot^3 x)$$

$$\frac{d^3y}{dx^3} = 2[-\operatorname{cosec}^2 x + 3\cot^2 x(-\operatorname{cosec}^2 x)]$$

$$= -2(1 + 3\cot^2 x)\operatorname{cosec}^2 x$$

$$= -2(1 + 3\cot^2 x)(1 + \cot^2 x)$$

$$= -2(1 + 4\cot^2 x + 3\cot^4 x)$$

(ii) $y = x^3 \sin ax$

Remarks

$$\frac{dy}{dx} = 3x^2.\sin ax + x^3.a\cos ax$$

$$\frac{d^2y}{dx^2} = 3[2x\sin ax + x^2.a\cos ax] + a[3x^2\cos ax + x^3(-a\sin ax)]$$

$$= (6x - a^2x^3)\sin ax + 6ax^2\cos ax$$

Example 2 : If $y = \sin(m\sin^{-1}x)$; prove that $(1 - x^2)y_2 - xy_1 + m^2y = 0$.

Solution :

$$y = \sin(m\sin^{-1}x)$$

$$y_1 = \cos(m\sin^{-1}x).\frac{m}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\,y_1 = m\cos(m\sin^{-1}x)$$

Squaring both sides, we have

$$(1 - x^2)y_1^2 = m^2\cos^2(m\sin^{-1}x)$$

$$= m^2[1 - \sin^2(m\sin^{-1}x)] = m^2 - m^2y^2$$

Differentiating both sides w.r.t. x, we have

$$(1 - x^2)\,2y_1\,y_2 - 2x\,y_1^2 = -m^2\,2yy_1$$

Dividing by $2y_1$, we get

$$(1 - x^2)\,y_2 - xy_1 + m^2y = 0.$$

Example 3 : If $x = at^2$, $y = 2at$, find $\frac{d^2y}{dx^2}$

Solution :

$$x = at^2 \quad , \quad y = 2at$$

$$\frac{dx}{dt} = 2at \quad , \quad \frac{dy}{dt} = 2a$$

Then

$$\frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx}$$

$$= 2a\times\frac{1}{2at} = \frac{1}{t}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{t^2}\frac{dt}{dx} = -\frac{1}{t^2}.\frac{1}{2at} = -\frac{1}{2at^3}$$

Example 4 : If $y = at^2$, $x = 2at$, find $\frac{d^2y}{dx^2}$

Solution :

$$y = at^2 \quad , \quad x = 2at$$

$$\frac{dy}{dt} = 2at \quad , \quad \frac{dx}{dt} = 2a$$

Now,

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$$

$$= 2at\times\frac{1}{2a}$$

$$\frac{dy}{dx} = t$$

Differentiating w.r.t. x

Remarks

$$\frac{d^2y}{dx^2} = 1.\frac{dt}{dx} = 1.\frac{1}{2a} = \frac{1}{2a}$$

Exercise 1.1

1. If $y = \frac{\log x}{x}$, show that $\frac{d^2y}{dx^2} = \frac{2\log x - 3}{x^3}$.
2. If y = A sin mx + B cos mx ; prove that $\frac{d^2y}{dx^2} + m^2y = 0$.
3. If $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$, find $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{2}$.
4. If y a cos(log x) + b sin(log x), show that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$.
5. If $y = (\sin^{-1}x)^2$ then prove that $(1 - x^2)y_2 - xy_1 = 2$.
6. If $p^2 = a^2\cos^2\theta + b^2\sin^2\theta$, prove that $p + \frac{d^2p}{d\theta^2} = \frac{a^2b^2}{p^3}$.
7. If $ax^2 + 2hxy + by^2 = 1$, show that $\frac{d^2y}{dx^2} = \frac{h^2 - ab}{(hx + by)^3}$.
8. If x = sin t, y = sin pt then prove that $(1 - x^2)y_2 - xy_1 + p^2y = 0$.
9. If $y = e^{\tan^{-1}x}$ then prove that $(1 + x^2)y_2 + (2x - 1)y_1 = 0$.
10. If $y = \left(x + \sqrt{1 + x^2}\right)^n$ then prove that $(x^2 + 1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - n^2y = 0$.
11. If $y = e^{m\sin^{-1}x}$ then prove that $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0$.
12. If $y = \left[\log(x + \sqrt{1 + x^2})\right]^2$ then prove that $(1 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 2 = 0$.
13. If $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$. Find the value of $\frac{d^2y}{dx^2}$ at x = a.
14. If $y = (\tan^{-1}x)^2$ then prove that $(x^2 + 1)^2y_2 + 2x(x^2 + 1)y_1 = 2$.
15. If x = 2cos t – cos 2t and y = 2sin t – sin 2t. Find $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{2}$.

1.3 n^{th} Derivatives

(i) To find the n^{th} derivative of x^m

Let $y = x^m$

$\therefore$ $y_1 = mx^{m-1}$

$y_2 = m(m - 1)x^{m-2}$

$y_3 = m(m - 1)(m - 2)x^{m-3}$

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$\therefore$ $y_n = m(m - 1)(m - 2)\ldots\ldots(m - n + 1)x^{m-n}$

Remark : If n = m, then

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$$y_m = m(m-1)(m-2)\ldots\ldots\ldots(m-n+1)\,x^{m-m}$$
$$= m(m-1)(m-2)\ldots 3.2.1. = m!$$
$$\mathbf{y_m = m!}$$

(ii) To find the n^{th} derivative of $(ax + b)^m$

Let $y = (ax+b)^m$

$\therefore$ $y_1 = ma\,(ax+b)^{m-1}$

$y_2 = m(m-1)\,a^2\,(ax+b)^{m-2}$

$y_3 = m(m-1)(m-2)\,a^3\,(ax+b)^{m-3}$

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$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\therefore$ $y_n = m(m-1)(m-2)\ldots.(m-n+1)\,a^n\,(ax+b)^{m-n}$

Remark : If n = m, then

$$y_m = m(m-1)(m-2)\ldots.1.a^n = m!\,a^m.$$

(iii) To find the n^{th} derivative of $\log(ax + b)$

Let $y = \log(ax+b)$

$\therefore$ $y_1 = \dfrac{1}{ax+b}.a = a(ax+b)^{-1}$

$y_2 = a^2(-1)(ax+b)^{-2}$

$y_3 = a^3(-1)(-2)(ax+b)^{-3}$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$y_n = a^n(-1)(-2)(-3)\ldots\ldots[-(n-1)]\,(ax+b)^{-n}$

$\therefore$ $y_n = \dfrac{(-1)(n-1)!a^n}{(ax+b)^n}$

(iv) To find the n^{th} derivative of $\dfrac{1}{ax+b}$, $ax + b \neq 0$

Let $y = \dfrac{1}{ax+b} = (ax+b)^{-1}$

$y_1 = -1(ax+b)^{-2}.a$

$y_2 = (-1)(-2)(ax+b)^{-3}.a^2$

$y_3 = (-1)(-2)(-3)(ax+b)^{-4}.a^3$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$y_n = (-1)(-2)(-3)\ldots..(-n)\,(ax+b)^{-(n+1)}.\,a^n$

$= (-1)^n.n!(ax+b)^{-(n+1)}.a^n$

$\therefore$ $y_n = \dfrac{(-1)^n n!a^n}{(ax+b)^{n+1}}.$

(v) To find the n^{th} derivative of $\cos(ax + b)$

Let $y = \cos(ax+b)$

$\therefore$ $y_1 = -a\sin(ax+b) = a\cos\left(ax+b+\dfrac{\pi}{2}\right)$ $\left[\because \cos\left(\dfrac{\pi}{2}+\theta\right) = -\sin\theta\right]$

$y_2 = -a^2\sin\left(ax+b+\dfrac{\pi}{2}\right) = a^2\cos\left(ax+b+\dfrac{2\pi}{2}\right)$

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$$y_3 = -a^3 \sin\left(ax + b + \frac{2\pi}{2}\right) = a^3 \cos\left(ax + b + \frac{3\pi}{2}\right)$$

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$$y_n = a^n \cos\left(ax + b + \frac{n\pi}{2}\right)$$

(vi) To find the n^{th} derivative of $e^{ax} \sin(bx + c)$

Let $\quad y = e^{ax} \sin(bx + c)$

$\therefore \quad y_1 = ae^{ax} \sin(bx + c) + be^{ax} \cos(bx + c)$

$\quad = e^{ax}[a \sin(bx + c) + b \cos(bx + c)]$

Put $\quad a = r \cos \alpha,\ b = r \sin \alpha$ so that $a^2 + b^2 = r^2$

$$\therefore \quad r = \sqrt{a^2 + b^2},\ \tan\alpha = \frac{b}{a} \Rightarrow \alpha = \tan^{-1}\frac{b}{a}$$

$$\begin{aligned}
\therefore \quad y_1 &= e^{ax}[\sin(bx + c) r \cos \alpha + \cos(bx + c)\, r \sin \alpha] \\
&= re^{ax}[\sin(bx + c) \cos \alpha + \cos(bx + c) \sin \alpha] \\
&= re^{ax} \sin(bx + c + \alpha) \\
y_2 &= r[ae^{ax} \sin(bx + c + \alpha) + e^{ax}\, b \cos(bx + c + \alpha)] \\
&= re^{ax} [a \sin(bx + c + \alpha) + b \cos(bx + c + \alpha)] \\
&= re^{ax}[r \cos \alpha \sin(bx + c + \alpha) + r \sin \alpha \cos(bx + c + \alpha)] \\
&= r^2e^{ax} [\sin(bx + c + \alpha) \cos \alpha + \cos(bx + c + \alpha) \sin \alpha] \\
&= r^2 e^{ax} \sin(bx + c + 2\alpha)
\end{aligned}$$

and so on.

In general $\quad y_n = r^n e^{ax} \sin(bx + c + n\alpha)$

$$\therefore \quad y_n = (a^2 + b^2)^{n/2} e^{ax} \sin\left(bx + c + n\tan^{-1}\frac{b}{a}\right)$$

Example 1 : Find n^{th} derivative of $\dfrac{1}{2x+3}$.

Solution : $\quad y = \dfrac{1}{2x+3}$

$$y_n = \frac{(-1)^n \underline{|n} 2^n}{(2x+3)^{n+1}} \qquad \left[\text{using } y = \frac{1}{ax+b} \text{ then } y_n = \frac{(-1)^n \underline{|n}\, a^n}{(ax+b)^{n+1}}\right]$$

Example 2 : Find n^{th} derivative of $e^{2x}\cos^2 x$.

Solution : $\quad y = e^{2x} \cos^2 x$

$$= e^{2x}\left(\frac{1+\cos 2x}{2}\right)$$

$$y = \frac{1}{2}e^{2x} + \frac{1}{2}e^{2x}\cos 2x$$

Then

$$y_n = \frac{1}{2}e^{2x}.2^n + \frac{1}{2}e^{2x}\left(2^2 + 2^2\right)^{\frac{n}{2}} \cos\left(2x + n\tan^{-1}(1)\right)$$

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$$= 2^{n-1}e^{2x} + \frac{1}{2}e^{2x}.2^{3\times\frac{n}{2}}\cos\left(2x + n\frac{\pi}{4}\right)$$

$$= 2^{n-1}e^{2x} + e^{2x}.2^{\frac{3n}{2}-1}\cos\left(2x + \frac{n\pi}{4}\right)$$

$$= 2^{n-1}e^{2x} + 2^{\frac{3n-2}{2}}e^{2x}\cos\left(2x + \frac{n\pi}{4}\right)$$

Example 3 : Find n^{th} derivative of $\sin^3 x \cos^3 x$.

Solution : $y = \sin^3 x \cos^3 x$

$$= \frac{1}{8}(8\sin^3 x\cos^3 x)$$

$$= \frac{1}{8}(2\sin x\cos x)^3$$

$$= \frac{1}{8}(\sin 2x)^3$$

$$y = \frac{1}{8}\sin^3 2x$$

$$y = \frac{1}{8}\left(\frac{3}{4}\sin 2x - \frac{1}{4}\sin 6x\right) \qquad \left[\begin{array}{l}\sin 3A = 3\sin A - 4\sin^3 A \\ 4\sin^3 A = 3\sin A - \sin 3A \\ \Rightarrow 4\sin^3 2x = 3\sin 2x - \sin 6x \\ \Rightarrow \sin^3 2x = \frac{3}{4}\sin 2x - \frac{1}{4}\sin 6x\end{array}\right]$$

$$y = \frac{3}{32}\sin 2x - \frac{1}{32}\sin 6x$$

$$y_n = \frac{3}{32}\sin\left(2x + \frac{n\pi}{2}\right).2^n - \frac{1}{32}\sin\left(6x + \frac{n\pi}{2}\right).6^n$$

1.4 USE OF PARTIAL FRACTIONS

(i) $$\frac{1}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$$

(ii) $$\frac{1}{(x-a)(x-b)(x-c)^3} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} + \frac{D}{(x-c)^2} + \frac{E}{(x-c)^3}$$

(iii) $$\frac{1}{(x-a)(x-b)(x^2+a^2)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{Cx+D}{x^2+a^2}$$

Example 1 : Find the n^{th} derivative of $\frac{x+1}{6x^2-7x-3}$

Solution : $y = \frac{x+1}{6x^2-7x-3}$

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$$= \frac{x+1}{6x^2 - 9x + 2x - 3} = \frac{x+1}{3x(2x-3) + 1(2x-3)}$$

$$y = \frac{x+1}{(3x+1)(2x-3)}$$

Let $\frac{x+1}{(3x+1)(2x-3)} = \frac{A}{3x+1} + \frac{B}{2x-3}$

$\Rightarrow$ $x + 1 = A(2x - 3) + B(3x + 1)$

Put $3x + 1 = 0$

$$x = -\frac{1}{3}$$

$$-\frac{1}{3} + 1 = A\left(-\frac{2}{3} - 3\right) \Rightarrow \frac{2}{3} = A\left(\frac{-2-9}{3}\right)$$

$$\frac{2}{3} = A\left(-\frac{11}{3}\right)$$

$$A = -\frac{2}{11}$$

Put $2x - 3 = 0$

$$x = \frac{3}{2}$$

$$\frac{3}{2} + 1 = B\left(3\left(\frac{3}{2}\right) + 1\right)$$

$$\frac{5}{2} = B\left(\frac{11}{2}\right)$$

$$B = \frac{5}{11}$$

$\Rightarrow$ $$y = -\frac{2}{11} \cdot \frac{1}{3x+1} + \frac{5}{11} \cdot \frac{1}{2x-3}$$

$$y_n = -\frac{2}{11} \frac{(-1)^n \lfloor n \, 3^n}{(3x+1)^{n+1}} + \frac{5}{11} \frac{(-1)^n \lfloor n \, 2^n}{(2x-3)^{n+1}}$$

Example 2 : If $y = \frac{x^2}{(x-1)^3 (x-2)}$; find y_n.

Solution : $$\frac{x^2}{(x-1)^3 (x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{x-2}$$

$\therefore$ $x^2 = A(x-1)^2(x-2) + B(x-1)(x-2) + c(x-2) + D(x-1)^3$ (1)

Putting $x = 1$ in (1), $1 = -C \Rightarrow$ $C = -1$

Putting $x = 2$ in (1), $4 = D$

Putting $x = 0$ in (1), $0 = -2A + 2B - 2C - D$

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or $2A - 2B = -2 \Rightarrow A - B = -1$ (2)

Putting $x = -1$ in (1) $1 = -12A + 6B - 3C - 8D$

or $12A - 6B = 3 - 32 - 1 = -30 \Rightarrow 2A - B = -5$ (3)

Solving (2) and (3), $A = -4$

$\therefore$ $B = A + 1 = -4 + 1 = -3$

Let $$y = -\frac{4}{x-1} - \frac{3}{(x-1)^2} - \frac{1}{(x-1)^3} + \frac{4}{x-2}$$

$\therefore$ $y_1 = -4(-1)(x-1)^{-2} - 3(-2)(x-1)^{-3} - (-3)(x-1)^{-4} + 4(-1)(x-2)^{-2}$

$y_2 = -4(-1)(-2)(x-1)^{-3} - 3(-2)(-3)(x-1)^{-4} - (-3)(-4)(x-1)^{-5} + 4(-1)(-2)(x-2)^{-3}$

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$y_n = -4(-1)(-2)\ldots(-n)(x-1)^{-(n+1)}\ (-3)(-2)(-3)\ldots[-(n+1)](x-1)^{-(n+2)}$

$-(-3)(-4)(-5)\ldots[-(n+2)]\ (x-1)^{-(n+3)} + 4(-1)(-2)\ldots(-n)(x-2)^{-(n+1)}$

$$= (-1)^n n!\left[-\frac{4}{(x-1)^{n+1}} - \frac{3(n+1)}{(x-1)^{n+2}} - \frac{(n+1)(n+2)}{2(x-1)^{n+3}} + \frac{4}{(x-2)^{n+1}}\right]$$

Exercise 1.2

Find the n^{th} derivatives of the following :

1. (i) $\frac{1}{(4-3x)^3}$ (ii) $\frac{1}{(3-2x)^3}$ (iii) $e^{3x} + e^{-3x}$ (iv) $\log 3x$

2. (i) $\log (x^2 - a^2)$ (ii) $\log\sqrt{\frac{2x+1}{x-2}}$ (iii) $\cos^4 x$

3. (i) $\cos ax \sin bx$ (ii) $\sin x \sin 2x \sin 3x$ (iii) $\cos x \cos 2x \cos 3x$

4. (i) $e^x \sin x \cos x$ (ii) $e^x \sin^3 x$

5. (i) $e^x \cos x \cos 2x$ (ii) $e^{3x} \sin^2 2x$ (iii) $e^x \cos^3 x$

6. Find the nth derivatives of the following :

(i) $\frac{1}{x^2 - a^2}$ (ii) $\frac{x^4}{(x-1)(x-2)}$ (iii) $\frac{x^2}{(x-1)^3(x+1)}$

7. Find the n^{th} derivative of

(i) $\frac{x^2}{(x-1)^3(x+1)}$ (ii) $\frac{1}{(x^2-a^2)(x^2-b^2)}$

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8. Find n^{th} derivative of $\sqrt{ax+b}$.

9. Find n^{th} derivative of $\log\sqrt{ax+x^2}$.

10. Find n^{th} derivative of $\sin^3 x \cos^3 x$.

Answers

1. (i) $\frac{3^n(n+2)!}{2!}.\frac{1}{(4-3x)^{n+3}}$ (ii) $\frac{(n+2)!2^{n-1}}{(3-2x)^{n+3}}$

(iii) $3^n[e^{3x}+(-1)^n e^{-3x}]$ (iv) $\frac{(-1)^{n-1}(n-1)!}{x^n}$

2. (i) $(-1)^{n-1}(n-1)!\left[\frac{1}{(x+a)^n}+\frac{1}{(x-a)^n}\right]$ (ii)

$\frac{(-1)^{n-1}(n-1)!}{2}\left[\frac{2^n}{(2x+1)^n}-\frac{1}{(x-2)^n}\right]$

(iii) $2^{n-1}\cos\left(2x+\frac{n\pi}{2}\right)+2^{2n-3}\cos\left(4x+\frac{n\pi}{2}\right)$

3. (i) $\frac{1}{2}\left[(a+b)^n\sin\left((a+b)x+\frac{n\pi}{2}\right)+(b-a)^n\sin\left((b-a)x+\frac{n\pi}{2}\right)\right]$

(ii) $\frac{1}{4}\left[2^n\sin\left(2x+\frac{n\pi}{2}\right)+4^n\sin\left(4x+\frac{n\pi}{2}\right)-6^n\sin\left(6x+\frac{n\pi}{2}\right)\right]$

(iii) $\frac{1}{4}\left[2^n\cos\left(2x+\frac{n\pi}{2}\right)+4^n\cos\left(4x+\frac{n\pi}{2}\right)+6^n\cos\left(6x+\frac{n\pi}{2}\right)\right]$

4. (i) $\frac{1}{2}.5^{\frac{n}{2}}e^x\sin(2x+n\tan^{-1}2)$

(ii) $\frac{3}{4}.2^{\frac{n}{2}}e^x\sin\left(x+\frac{n\pi}{4}\right)-\frac{1}{4}.10^{\frac{n}{2}}e^x\sin(3x+n\tan^{-1}3)$

5. (i) $\frac{1}{2}e^x\left[2^{\frac{n}{2}}\cos\left(x+\frac{n\pi}{4}\right)+10^{\frac{n}{2}}\cos(3x+n\tan^{-1}3)\right]$

(ii) $\frac{1}{2}e^{3x}\left[3^n-5^n\cos\left(4x+n\tan^{-1}\frac{4}{3}\right)\right]$

(iii) $\frac{3}{4}.2^{\frac{n}{2}}e^x\cos\left(x+\frac{n\pi}{4}\right)+\frac{1}{4}.10^{\frac{n}{2}}e^x\cos(3x+n\tan^{-1}3)$

6. (i) $\frac{1}{2a}(-1)^n n!\left[\frac{1}{(x-a)^{n+1}}-\frac{1}{(x+a)^{n+1}}\right]$

(ii) $(-1)^n n!\left[\frac{16}{(x-2)^{n+1}}-\frac{1}{(x-1)^{n+1}}\right]$; $n>2$

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(iii) $(-1)^n n!\left[\frac{1}{8(x-1)^{n+1}}+\frac{3(n+1)}{4(x-1)^{n+2}}+\frac{(n+1)(n+2)}{4(x-1)^{n+3}}-\frac{1}{8(x+1)^{n+1}}\right]$

7. (i) $(-1)^n n!\left[\frac{(n+2)(n+1)}{4(x-1)^{n+3}}+\frac{3(n+1)}{4(x-1)^{n+2}}+\frac{1}{8(x-1)^{n+1}}-\frac{1}{8(x+1)^{n+1}}\right]$

(ii) $\frac{(-1)^n n!}{a^2-b^2}\left[\frac{1}{2a}\left(\frac{1}{(x-a)^{n+1}}-\frac{1}{(x+a)^{n+1}}\right)-\frac{1}{2b}\left(\frac{1}{(x-b)^{n+1}}-\frac{1}{(x+b)^{n+1}}\right)\right]$

8. $y_n=(-1)^{n-1}\frac{1.3.5...(2n-3)}{2^n}\frac{a^n}{(ax+b)^{\frac{2n-1}{2}}}$

9. $y_n=\frac{(-1)^{n-1}\lfloor n-1}{2}\left[\frac{1}{x^n}+\frac{1}{(x+a)^n}\right]$

10. $y_n=\frac{1}{8}\cos\left(x+\frac{n\pi}{2}\right)-\frac{5^n}{16}\cos\left(5x+\frac{n\pi}{2}\right)-\frac{3^n}{16}\cos\left(3x+\frac{n\pi}{2}\right).$

1.5 SOME MORE n^{th} DERIVATIVES

Example 1 : Find n^{th} derivative of $\cot^{-1}\frac{x}{a}$.

Solution : Let $y=\cot^{-1}\frac{x}{a}$

$$y_1=\frac{-a}{a^2+x^2}$$

$$\left[\text{Put } y=\cot^{-1}\frac{x}{a}\Rightarrow \cot y=\frac{x}{a}\Rightarrow x=a\cot y\right]$$

$$y_1=\frac{-a}{a^2(1+\cot^2 y)}=-\frac{1}{a}\sin^2 y$$

$$y_2=-\frac{1}{a}(2\sin y\cos y).y_1$$

$$=-\frac{1}{a}(\sin 2y)\left(-\frac{\sin^2 y}{a}\right)$$

$$y_2=\frac{(-1)^2}{a^2}\sin 2y.\sin^2 y$$

Similarly, $y_3=\frac{(-1)^3}{a^3}\lfloor 2\sin^3 y\sin 3y$

....

$y_n=\frac{(-1)^n}{a^n}\lfloor n-1\sin^n y\sin ny$ where $y=\cot^{-1}\frac{x}{a}$

Remarks

Exercise 1.3

1. $\tan^{-1}\dfrac{x}{a}$ 2. $\tan^{-1}\dfrac{2x}{1-x^2}$ 3. $\tan^{-1}\dfrac{1+x}{1-x}$

4. $\dfrac{1}{x^2+x+1}$ 5. $\dfrac{1}{x^4-a^4}$ 6. $\cot^{-1} x$

7. $\dfrac{1}{x^2+a^2}$ 8. $\dfrac{x}{x^2+a^2}$

Answers

1. $\dfrac{(-1)^{n-1}(n-1)!}{a^n}\sin n\left(\dfrac{\pi}{2}-y\right)\sin^n\left(\dfrac{\pi}{2}-y\right)$ where $y=\tan^{-1}\dfrac{x}{a}$

2. $2(-1)^{n-1}(n-1)!\ \sin n\theta\ \sin^n\theta$, where $\theta=\cot^{-1} x$

3. $(-1)^{n-1}(n-1)!\ \sin n\theta\ \sin^n\theta$, where $\theta=\cot^{-1} x$

4. $\left(\dfrac{2}{\sqrt{3}}\right)^{n+2}(-1)^n n!\sin(n+1)\theta\sin^{n+1}\theta$ where $\theta=\cot^{-1}\dfrac{2x+1}{\sqrt{3}}$

5. $\dfrac{(-1)^n n!}{4a^3}\left[\dfrac{1}{(x-a)^{n+1}}-\dfrac{1}{(x+a)^{n+1}}-\dfrac{2}{a^{n+1}}\sin(n+1)\theta\sin^{n+1}\theta\right]$, where $\theta=\cot^{-1}\dfrac{x}{a}$.

6. $y_n=(-1)^n\underline{|n-1}\sin ny\sin^n y$ where $y=\cot^{-1}x$.

7. $y_n=\dfrac{(-1)^n\underline{|n}}{a^{n+2}}\sin(n+1)\theta\sin^{n+1}\theta$ where $\theta=\cot^{-1}\dfrac{x}{a}$.

8. $y_n=\dfrac{(-1)^n\underline{|n}}{a^{n+1}}\cos(n+1)\theta\sin^{n+1}\theta$ where $\theta=\cot^{-1}\dfrac{x}{a}$.

1.6 LEIBNITZ'S THEOREM

To find the n^{th} derivative of the product of two functions we employ Leibnitz's theorem which is given below :

Statement : If u, v are functions of x possessing derivatives upto n^{th} order, then

$$(uv)_n=u_nv+{}^nC_1u_{n-1}v_1+{}^nC_2u_{n-2}v_2+{}^nC_3u_{n-3}v_3+\ldots..+{}^nC_ru_{n-r}v_r+\ldots+{}^nC_nuv_n$$

Where the suffixes of u and v denote the order of the derivatives.

Proof : We shall prove the result by Mathematical Induction.

Step I. We know that $(uv)_1=u_1v+uv_1$

$$(uv)_2=u_2v+u_1v_1+u_1v_1+uv_2$$
$$=u_2v+{}^2C_1u_1v_1+{}^2C_2uv_2$$

This shows that the theorem is true for n = 1, 2

Remarks

Step II. Suppose the theorem is true for n = m

$$\therefore \quad (uv)_m = u_m v + {}^mC_1 u_{m-1}v_1 + {}^mC_2 u_{m-2}v_2 + {}^mC_3 u_{m-3}v_3 + \ldots + {}^mC_{r-1}u_{m-r+1}v_{r-1}$$

$$+ {}^mC_r u_{m-r}v_r + \ldots\ldots + {}^mC_{m-1}u_1 v_{m-1} + {}^mC_m uv_m + \qquad (1)$$

Step III. Differentiating both sides of (1), we have

$$(uv)_{m+1} = u_{m+1}v + u_m v_1 + {}^mC_1(u_m v_1 + u_{m-1}v_2) + {}^mC_2(u_{m-1}v_2 + u_{m-2}v_3)$$

$$+ {}^mC_3(u_{m-2}v_3 + u_{m-3}v_4) + \ldots. + {}^mC_{r-1}(u_{m-r+2}v_{r-1} + u_{m-r+1}v_r)$$

$$+ {}^mC_r(u_{m-r+1}v_r + u_{m-r}v_{r+1}) + \ldots. + {}^mC_{m-1}(u_2 v_{m-1} + u_1 v_m) + {}^mC_m(u_1 v_m + uv_{m+1})$$

$$= u_{m+1}v + (1 + {}^mC_1)u_m v_1 + ({}^mC_1 + {}^mC_2)u_{m-1}v_2 + ({}^mC_2 + {}^mC_3)u_{m-2}v_3 + \ldots.$$

$$+ ({}^mC_{r-1} + {}^mC_r)u_{m-r+1}v_r + \ldots + ({}^mC_{m-1} + {}^mC_m)u_1 v_m + {}^mC_m uv_{m+1}$$

$$= u_{m+1}v + {}^{m+1}C_1 u_m v_1 + {}^{m+1}C_2 u_{m-1}v_2 + {}^{m+1}C_3 u_{m-2}v_3 + \ldots. + {}^{m+1}C_r u_{m-r+1}v_r + \ldots$$

$$+ {}^{m+1}C_m u_1 v_m + {}^{m+1}C_{m+1}uv_{m+1}$$

$$\because\ 1 + {}^mC_1 = {}^mC_0 + {}^mC_1 = {}^{m+1}C_1$$

$${}^mC_1 + {}^mC_2 = {}^{m+1}C_2$$

$${}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r$$

$${}^mC_m = 1 = {}^{m+1}C_{m+1}$$

Hence result is true for n = m + 1 by Induction Method, the result is true for all n.

Example 1 : Find n^{th} derivative of $e^x \log x$

Solution : $y = e^x \log x$

Differentiating n times by Leibnitz's Theorem

$$y_n = {}^nC_0 \log x.e^x + {}^nC_1 \frac{1}{x}e^x + {}^nC_2\left(\frac{-1}{x^2}\right)e^x + \ldots.. + {}^nC_n \frac{(-1)^{n-1}\underline{|n-1}}{x^n}.e^x$$

$$= \log x.e^x + n.\frac{1}{x}e^x + \frac{n(n-1)}{\underline{|2}}\left(\frac{-1}{x^2}\right)e^x + \ldots\ldots + \frac{(-1)^{n-1}\underline{|n-1}}{x^n}.e^x$$

Example 2 : If $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$, prove that

$$(x^2 - 1)\, y_{n+2} + (2n + 1)\, xy_{n+1} + (n^2 - m^2)\, y_n = 0.$$

Solution : $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$

$$\therefore \quad y^{\frac{1}{m}} + \frac{1}{y^{\frac{1}{m}}} = 2x$$

(1)

Remarks

Let $y^{\frac{1}{m}} = t$, then from (1), we have

$\therefore$ $$t + \frac{1}{t} = 2x$$

or $t^2 - 2xt + 1 = 0 \Rightarrow$ $$t = \frac{2x \pm \sqrt{4x-4}}{2} = x \pm \sqrt{x^2-1}$$

$\therefore$ $$y^{\frac{1}{m}} = x \pm \sqrt{x^2-1}$$

$\Rightarrow$ $$y = (x + \sqrt{x^2-1})^m \quad \text{or} \quad (x - \sqrt{x^2-1})^m$$

If $y = (x + \sqrt{x^2-1})^m$, then

$$y_1 = m(x + \sqrt{x^2-1})^{m-1}\left[1 + \frac{2x}{2\sqrt{x^2-1}}\right]$$

$$= m(x + \sqrt{x^2-1})^{m-1}\left[\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}\right]$$

$$= m\frac{(x + \sqrt{x^2-1})^m}{\sqrt{x^2-1}} = \frac{my}{\sqrt{x^2-1}}$$

i.e., $$\sqrt{x^2-1}\; y_1 = my$$

(2)

Similarly, if $y = (x - \sqrt{x^2-1})^m$, then

$$y_1 = -\frac{my}{\sqrt{x^2-1}}$$

(3)

i.e., $$\sqrt{x^2-1}\; y_1 = -my$$

From (2) and (3), we have

$$(x^2-1)y_1^{\,2} = m^2y^2$$

(4)

Differentiating both sides of (4) w.r.t x, we have

$$(x^2-1)2y_1y_2 + 2xy_1^{\,2} = m^2 2yy_1$$

or $$(x^2-1)y_2 + xy_1 - m^2y = 0 \qquad (5)$$ (Dividing both sides by $2y_1$)

Differentiating (5) w.r.t. x, n times by Leibnitz's theorem, we have

$$(y_2)_n(x^2-1) + {}^nC_1(y_2)_{n-1}(2x) + {}^nC_2(y_2)_{n-2}.2 + [(y_1)_n x + {}^nC_1(y_1)_{n-1}.1] - m^2y_n = 0$$

$$y_{n+2}(x^2-1) + ny_{n+1}2x + \frac{n(n-1)}{2!}y_n.2 + y_{n+1}x + ny_n.1 - m^2y_n = 0$$

$\therefore$ $$(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0.$$

Example 3 : If $y = x^2 \quad e^x$, then prove that

$$y_n = \frac{n(n-1)}{2}y_2 - n(n-2)y_1 + \frac{(n-1)(n-2)y}{2}$$

Remarks

Solution : $y = x^2 e^x$

$$y_n = {}^nC_0 x^2 e^x + {}^nC_1(2x)e^x + {}^nC_2(2)e^x$$

$$= x^2e^x + n(2x)e^x + \frac{n(n-1)(2)e^x}{2}$$

$$y_n = e^x(x^2 + 2nx + n(n-1))$$

Putting n = 1, 2

$$y_1 = e^x(x^2 + 2x)$$

$$y_2 = e^x(x^2 + 4x + 2)$$

R.H.S. $\frac{n(n-1)}{2}y_2 - n(n-2)y_1 + \frac{(n-1)(n-2)y}{2}$

$$= \frac{n(n-1)}{2}\left(e^x\left(x^2 + 4x + 2\right)\right) - n(n-2)e^x(x^2+2x) + \frac{(n-1)(n-2)}{2}x^2e^x$$

$$= e^x(x^2 + 2nx + n(n-1))$$

Hence L.H.S. = R.H.S.

Example 4 : If $y = e^{m\sin^{-1}x}$ then prove that

$$(1 - x^2)y_{n+2} - (2n + 1)\, xy_{n+1} - (n^2 + m^2)y_n = 0$$

Solution : $y = e^{m\sin^{-1}x}$

$$y_1 = e^{m\sin^{-1}x}.\frac{m}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}.y_1 = e^{m\sin^{-1}x}.m$$

$\Rightarrow$ $\sqrt{1-x^2}.y_1 = my$

Taking square both sides

$$(1 - x^2)y_1^2 = m^2 y^2$$

Differentiating w.r.t. x, we get

$$(1 - x^2)2y_1y_2 + y_1^2(-2x) = m^2(2yy_1)$$

$$2y_1((1 - x^2)y_2 - xy_1) = 2y_1(m^2y)$$

$\Rightarrow$ $(1 - x^2)y_2 - xy_1 - m^2y = 0$

Differentiating n times by Leibnitz's Theorem

$$\left[{}^nC_0(1-x^2)y_{n+2} + {}^nC_1(-2x)y_{n+1} + {}^nC_2(-2)y_n\right] - \left[{}^nC_0xy_{n+1} + {}^nC_1(1)y_n\right] - m^2y_n = 0$$

$\Rightarrow$ $(1-x^2)y_{n+2} - 2nxy_{n+1} - \frac{n(n-1)}{2}.2y_n - xy_{n+1} - ny_n - m^2y_n = 0$

$\Rightarrow$ $(1 - x^2)y_{n+2} - (2n + 1)\, xy_{n+1} - (n^2 + m^2)y_n = 0$

Exercise 1.4

1. Find the n^{th} derivatives of the following :
   (i) $x^3 \log x$ (ii) $x^3 \sin ax$ (iii) $e^x \sin x \sin 2x$ (iv) $x^2 e^{ax} \sin bx$

2. Find the n^{th} derivative of $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + a^2y = 0$.

3. If $y = x^n \log x$, show that $y_{n+1} = \frac{n!}{x}$.

Remarks

4. If y = a cos (log x) + b sin (log x), prove that
$$x^2 y_{n+2} + (2n + 1) xy_{n+1} + (n^2 + 1) y_n = 0.$$
5. If $y = \tan^{-1}x$, prove that $(1 + x^2)y_{n+1} + 2nxy_n + n(n + 1)y_{n-1} = 0$.
6. If $y = \sin^{-1}x$, prove that $(1 – x^2)y_{n+2} – (2n + 1)xy_{n+1} – n^2y_n = 0$.
7. If $y = e^{m\cos^{-1}x}$, prove that $(1 – x^2)y_{n+2} – (2n + 1)xy_{n+1} – (n^2 + m^2)y_n = 0$.
8. If $y = (x^2 – 1)^n$, prove that $(x^2 – 1)y_{n+2} + 2xy_{n+1} – n(n + 1)y_n = 0$.
9. If $x = \sin\left(\frac{\log y}{a}\right)$, prove that $(1 – x^2)y_{n+2} – (2n + 1)xy_{n+1} – (n^2 + a^2)y_n = 0$.
10. If $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$, prove that $x^2y_{n+2} + (2n + 1)xy_{n+1} + 2n^2y_n = 0$.
11. If $y = \sin(m \sin^{-1} x)$ then prove that $(1 – x^2)y_{n+2} – (2n + 1)xy_{n+1} + (m^2 – n^2)y_n = 0$.
12. If $y = \left(\log\left(x + \sqrt{1 + x^2}\right)\right)^2$ then prove that $(1 + x^2)y_{n+2} + (2n + 1)xy_{n+1} + n^2y_n = 0$.

Answers

1. (i) $\frac{(-1)^{n-1}n!}{x^{n-3}}\left\{\frac{1}{n} - \frac{3}{n-1} + \frac{3}{n-2} - \frac{1}{n-3}\right\}$ if $n > 3$

(ii) $$x^3a^n \sin\left(ax + \frac{n\pi}{2}\right) + 3nx^2a^{n-1}\sin\left(ax + (n-1)\frac{\pi}{2}\right) + 3n(n-1)xa^{n-2}\sin\left(ax + (n-2)\frac{\pi}{2}\right)$$
$$+n(n-1)(n-2)a^{n-3}\sin\left(ax + (n-3)\frac{\pi}{2}\right)$$

(iii) $$\frac{1}{2}e^x\left[\cos x + {}^nC_1\cos\left(x + \frac{\pi}{2}\right) + {}^nC_2\cos\left(x + \frac{2\pi}{2}\right) + ... + {}^nC_n\cos\left(x + \frac{n\pi}{2}\right)\right]$$
$$-\frac{1}{2}e^x\left[\cos 3x + 3\,{}^nC_1\cos\left(3x + \frac{\pi}{2}\right) + 3^2\,{}^nC_2\cos\left(3x + \frac{2\pi}{2}\right) + ... + 3^n\,{}^nC_n\cos\left(3x + \frac{n\pi}{2}\right)\right]$$

(iv) $$x^2(a^2 + b^2)^{\frac{n}{2}}e^{ax}\sin\left(bx + n\tan^{-1}\frac{b}{a}\right) + 2nx(a^2 + b^2)^{\frac{(n-1)}{2}}e^{ax}\sin\left(bx + (n-1)\tan^{-1}\frac{b}{a}\right)$$
$$+n(n-1)(a^2 + b^2)^{\frac{(n-2)}{2}}e^{ax}\sin\left(bx + (n-2)\tan^{-1}\frac{b}{a}\right)$$

2. $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - (n^2 - a^2)y_n = 0.$

1.7 CALCULATION OF ‘n^{th}’ DERIVATIVE AT x = 0

Even though sometimes it is difficult to calculate the n^{th} derivative of a function in general, its value can be calculated at x = 0 by the application of Leibnitz’s theorem.

Example 1 : If $y = \frac{\sin^{-1}x}{\sqrt{1 - x^2}}$, then prove that $y_n(0) = (n – 1)^2 y_{n-2}(0)$.

Remarks

Solution : $y = \dfrac{\sin^{-1} x}{\sqrt{1-x^2}}$

$\Rightarrow$ $\sqrt{1-x^2}\; y = \sin^{-1} x$

Differentiating w.r.t. x

$$\sqrt{1-x^2}\; y_1 + y.\frac{1}{2\sqrt{1-x^2}}(-2x) = \frac{1}{\sqrt{1-x^2}}$$

$\Rightarrow$ $(1 - x^2)y_1 - xy_1 = 1$

Differentiating n times by Leibnitz's Theorem

$$\left[{}^nC_0(1-x^2)y_{n+1} + {}^nC_1(-2x)y_n + {}^nC_2(-2)y_{n-1}\right] - \left[{}^nC_0 xy_{n+1} + {}^nC_1(1)y_n\right] = 0$$

$\Rightarrow$ $(1 - x^2)y_{n+1} - (2n + 1)xy_n - n^2\, y_{n-1} = 0$

Putting x = 0

$\Rightarrow$ $y_{n+1}(0) = n^2 y_{n-1}(0)$

Change n to n – 1, we get

$y_n(0) = (n - 1)^2 y_{n-2}(0)$

Hence proved.

Example 2 : If $y = e^{m\cos^{-1} x}$, show that $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - (n^2 + m^2)y_n = 0$ and find the value of $y_n(0)$.

Solution : $y = e^{m\cos^{-1} x}$ (1)

$\therefore$ $y_1 = e^{m\cos^{-1} x}.\dfrac{-m}{\sqrt{1-x^2}}$ (2)

Cross-multiplying and squaring, we get

$$(1-x^2){y_1}^2 = m^2\left(e^{m\cos^{-1} x}\right)^2 = m^2y^2 \quad \text{[By (1)]}$$

Differentiating both sides w.r.t. x, we have

$$(1 - x^2).\,2y_1y_2 + {y_1}^2.(-2x) = m^2.2yy_1$$

Dividing by $2y_1$, we get

$$(1 - x^2)y_2 - xy_1 = m^2y \quad (3)$$

Differentiating n-times by Leibnitz's theorem, we have

$$\left[y_{n+2}(1-x^2) + ny_{n+1}(-2x) + \frac{n(n-1)}{2!}y_n(-2)\right] - \left[y_{n+1}.x + ny_n.1\right] = m^2y_n$$

$$(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - (n^2 + m^2)y_n = 0 \quad (4)$$

When x = 0, from (1), (2), (3) and (4), we get

$$y(0) = e^{\frac{m\pi}{2}}$$

$$y_1(0) = -me^{\frac{m\pi}{2}}$$

$$y_2(0) = m^2.y(0) = m^2.e^{\frac{m\pi}{2}}$$

$$y_{n+2}(0) = (n^2 + m^2)y_n(0) \quad (5)$$

Putting n = 1, 2, 3, 4, … in (5), we get

Remarks

$$y_3(0) = (1^2 + m^2)y_1(0) = -m(1^2 + m^2)e^{\frac{m\pi}{2}}$$

$$y_4(0) = (2^2 + m^2)y_2(0) = m^2(2^2 + m^2)e^{\frac{m\pi}{2}}$$

$$y_5(0) = (3^2 + m^2)y_3(0) = -m(1^2 + m^2)(3^2 + m^2)e^{\frac{m\pi}{2}}$$

$$y_6(0) = (4^2 + m^2)y_4(0) = m^2(2^2 + m^2)(4^2 + m^2)e^{\frac{m\pi}{2}}$$

Hence $y_n(0) = -m.e^{\frac{m\pi}{2}}(1^2 + m^2)(3^2 + m^2)....\left[(n-2)^2 + m^2\right]$ when n is odd

and $y_n(0) = m^2.e^{\frac{m\pi}{2}}(2^2 + m^2)(4^2 + m^2)....\left[(n-2)^2 + m^2\right]$ when n is even .

Exercise 1.5

1. If $y = \tan^{-1}x$, prove that $(1 + x^2)y_{n+1} + 2nxy_n + n(n - 1)y_{n-1} = 0$ and hence determine the value of $y_n(0)$.
2. If $y = \left[x + \sqrt{1+x^2}\right]^m$, find $y_n(0)$.
3. If $y = \left[\log\left(x + \sqrt{1+x^2}\right)\right]^2$, find $y_n(0)$.
4. If $y = e^{a\sin^{-1}x}$, prove that $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - (n^2 + a^2)y_n = 0$. Deduce that $\lim\limits_{x\to 0}\frac{y_{n+2}}{y_n} = n^2 + a^2$. Hence find $y_n(0)$.
5. If $y = \left(\sin^{-1}x\right)^2$, then prove that
   (i) $(1 - x^2)y_2 - xy_1 - 2 = 0$
   (ii) $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - n^2y_n = 0$
   Hence deduce that $\lim\limits_{x\to 0}\frac{y_{n+2}}{y_n} = n^2$ and find $y_n(0)$.

Answers

1. $y_n(0) = 0$, if n is even and $y_n(0) = (-1)^{\frac{n-1}{2}}(n-1)!$ if n is odd.
2. $y_n(0) = m^2(m^2 - 2^2)(m^2 - 4^2)\ldots[m^2 - (n - 2)^2]$ if n is even.
   $y_n(0) = m(m^2 - 1^2)(m^2 - 3^2)\ldots.[m^2 - (n - 2)^2]$ if n is odd.
3. $y_n(0) = 0$ if n is odd and $y_n = (-1)^{\frac{n}{2}-1}2.2^2.4^2...(n-2)^2$ if n is even.
4. $y_n(0) = a^2(2^2 + a^2)(4^2 + a^2)\ldots.[(n - 2)^2 + a^2]$ if n is even.
   $y_n(0) = a(1^2 + a^2)(3^2 + a^2)\ldots.[(n - 2)^2 + a^2]$ if n is odd.

Keywords : Successive differentiation, n^{th} derivatives, Leibnitz theorem.

Summary

If $y = f(x)$, then successive derivatives are denoted by $y_1, y_2, y_3\ldots y_n$.

n^{th} derivative can be found by splitting into partial fraction and Leibnitz theorem. Even though sometimes it is difficult to calculate the n^{th} derivative of a function in general, its value can be calculated at x = 0 by application of Leibnitz theorem.

Remarks

CHAPTER – II

SOME GENERAL THEOREMS ON DIFFERENTIABLE FUNCTIONS AND EXPANSIONS

2.0 STRUCTURE

2.1 Introduction
2.2 Objective
2.3 Taylor's Theorem with Lagrange's Form of Remainder after 'n' Terms
2.4 Taylor's Theorem with Cauchy's Form of Remainder
2.5 Taylor's Infinite Series
2.6 Expansion by Differential Equations
2.7 Asymptotes
2.8 Intersection of the Curve and Its Asymptotes
2.9 Working Rule for finding Asymptotes of Polar Curves

2.1 INTRODUCTION

The students have already learnt the application of Rolle's theorem and LMVT (Lagrange mean value theorem), Tayloy's theorem can be regarded as a general form of LMVT when the function is differentiable n times, n > 1 and Rolle's theorem is used to prove Taylor's theorem. Let P be any point on y = f(x). Then moves further and further away from the origin, it may happen that distance between and some fixed line $\rightarrow$ 0. The line is called asymptote.

2.2 OBJECTIVE

After reading this lesson, you must be able to understand

- Taylor's theorem
- Maclamin's theorem
- Expression of trig. Functions, Log function & exponential function.
- Expansion by differentiation and integration
- Asymptotes

2.3 TAYLOR'S THEOREM WITH LAGRANGE'S FORM OF REMAINDER AFTER 'n' TERMS

If a function f(x) is such that

(i) $f(x), f'(x), f''(x)\ldots f^{n-1}(x)$ are continuous in closed interval [a, a + h]

(ii) $f''(x)$ exists in open interval (a, a + h),

then there exists atleast one real number θ; $0 <\theta< 1$ such that

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a + \theta h).$$

Proof : Consider the function

$$F(x) = f(x) + (a+h-x)f'(x) + \frac{(a+h-x)^2}{2!} f''(x) + \ldots..$$

$$\ldots + \frac{(a+h-x)^{n-1}}{(n-1)!} f^{n-1}(x) + \frac{(a+h-x)^n}{n!} .A \qquad (1)$$

where A is a constant to be chosen so that $\qquad F(a) = F(a + h) \qquad (2)$

Remarks

Now $F(a) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + ... + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n}{n!}A$ [By (1)]

and $F(a + h) = f(a + h)$

Putting these values in (2), we have

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n}{n!}A \quad (3)$$

Now

(i) $f(x), f'(x), f''(x)\ldots f^{n-1}(x)$ are continuous on the closed interval $[a, a + h]$…(given) and $(a + h - x)$, $(a + h - x)^2, (a + h - x)^3, \ldots (a + h - x)^n$ being polynomials are continuous on the closed interval $[a, a + h]$.

Also the algebraic sum of continuous functions is continuous

$\therefore$ F(x) is continuous in the closed interval $[a, a + h]$.

(ii) As $f''(x)$ exists in the open interval $(a, a + h)$

$\therefore$ $f(x), f'(x), f''(x), \ldots, f^{n-1}(x)$ are all derivable in the open interval $(a, a + h)$.

Also $(a + h - x), (a + h - x)^2, \ldots, (a + h - x)^n$ being polynomials are derivable in the open interval $(a, a + h)$.

F(x) is derivable in the open interval $(a, a + h)$.

(iii) $F(a) = F(a + h)$

$\therefore$ F(x) satisfies all the three conditions of Rolle's theorem in $[a, a + h]$. By Rolle's Theorem hence there exists atleast one real number θ; $0 < \theta < 1$, such that

$$F'(a + \theta h) = 0 \quad (4)$$

Differentiating (1) w.r.t. x, we get

$$\begin{aligned} F'(x) = f'(x) & \\ & + (a + h - x) f''(x) - f'(x) \\ & + \frac{(a+h-x)^2}{2!}f'''(x) + \frac{2(a+h-x)}{2!}(-1)f''(x) \\ & + \ldots\ldots\ldots\ldots\ldots\ldots \\ & + \ldots\ldots\ldots\ldots\ldots\ldots \\ & + \frac{(a+h-x)^{n-1}}{(n-1)!}f^n(x) - \frac{(n-1)(a+h-x)^{n-2}}{(n-1)!}f^{n-1}(x) \\ & + \frac{n(a+h-x)^{n-1}(-1)A}{n!} \end{aligned}$$

or
$$F'(x) = \frac{(a+h-x)^{n-1}}{(n-1)!}f^n(x) - \frac{(a+h-x)^{n-1}}{(n-1)!}A$$
$$= \frac{(a+h-x)^{n-1}}{(n-1)!}\left[f^n(x) - A\right]$$

Putting $x = a + \theta h$, we get

$$F'(a + \theta h) = \frac{\left[h(1-\theta)\right]^{n-1}}{(n-1)!}\left[f^n(a + \theta h) - A\right]$$

But $F'(a + \theta h) = 0$ [From (4)]

$\therefore$ $f^n(a + \theta h) - A = 0 \quad \Rightarrow \quad A = f^n(a + \theta h)$ $[\because 1 - \theta \neq 0; h \neq 0]$

Remarks

$\therefore$ From (2), $f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a + \theta h)$

Which is the required expression.

Note : The (n + 1)th term i.e. $\frac{h^n}{n!} f^n(a + \theta h)$ is called the **Lagrange's remainder after n terms and is denoted by R_n.**

Cor. Maclaurin's Theorem with Lagrange's form of remainder

Putting a = 0, h = x in Taylor's theorem, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n}{n!} f^n(\theta x) \qquad [0 < \theta < 1]$$

Which is Maclaurin's theorem with Lagrange's form of remainder.

2.4 TAYLOR'S THEOREM WITH CAUCHY'S FORM OF REMAINDER

If a function f(x) is such that

(i) $f(x), f'(x), f''(x) \ldots f^{n-1}(x)$ are continuous in closed interval [a, a + h]

(ii) $f^n(x)$ exists in open interval (a, a + h)

then there exists atleast one real number θ; $0 < \theta < 1$ such that

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{(n-1)!} (1-\theta)^{n-1} f^n(a + \theta h).$$

Proof : Let $F(x) = f(x) + (a + h - x) f'(x) + \frac{(a+h-x)^2}{2!} f''(x) + \ldots$

$$\ldots + \frac{(a+h-x)^{n-1}}{(n-1)!} f^{n-1}(x) + (a + h - x)A \qquad (1)$$

where A is a constant such that F(a) = F(a + h).

Putting x = a in (1), we have

$$F(a) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + hA \qquad (2)$$

Putting x = a + h in (1), we have

$$F(a + h) = f(a + h) + 0 + 0 + \ldots = f(a + h) \qquad (3)$$

Now F(a + h) = F(a)

Then from (2) and (3), we have

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + hA \qquad (4)$$

Now, (i) $f(x), f'(x), f''(x) \ldots f^{n-1}(x)$ are continuous on the closed interval [a, a + h]…(given) and $(a + h - x), (a + h - x)^2, (a + h - x)^3, \ldots (a + h - x)^n$ being polynomials are continuous on the closed interval [a, a + h].

Also the algebraic sum of continuous functions is continuous.

$\therefore$ F(x) is continuous in the closed interval [a, a + h].

(ii) As $f^n(x)$ exists in the open interval (a, a + h)

$\therefore$ $f(x), f'(x), f''(x), \ldots, f^{n-1}(x)$ are all derivable in the open interval (a, a + h).

Also $(a + h - x), (a + h - x)^2, \ldots, (a + h - x)^n$ being polynomials are derivable in the open interval (a, a + h).

$\therefore$ F(x) is derivable in the open interval (a, a + h).

(iii) F(a) = F(a + h)

Remarks

$\therefore$ F(x) satisfies all the three conditions of Rolle's theorem in [a, a + h]. By Rolle's Theorem hence there exists atleast one real number θ; $0 < \theta < 1$, such that

$$F'(a + \theta h) = 0$$

Differentiating both sides of (1) w.r.t. x,

$$F'(x) = f'(x) + [(a + h - x) f''(x) - f'(x)] + \frac{1}{2!}[(a + h - x)^2 f'''(x) - 2(a + h - x)f''(x)]$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\frac{1}{(n-1)!}[(a + h - x)^{n-1}f^{n}(x) - (n - 1)(a + h - x)^{n-2}f^{n-1}(x)] - A$$

or $$F'(x) = \frac{1}{(n-1)!}[(a + h - x)^{n-1}f^{n}(x) - A$$

Putting $x = a + \theta h$, we get

$$F'(a + \theta h) = \frac{1}{(n-1)!}(a + h - a - \theta h)^{n-1} f^{n}(a + \theta h) - A$$

$$= \frac{1}{(n-1)!}(h - \theta h)^{n-1} f^{n}(a + \theta h) - A$$

$$= \frac{h^{n-1}}{(n-1)!}(1-\theta)^{n-1} f^{n}(a + \theta h) - A$$

But $F'(a + \theta h) = 0$

$$\Rightarrow \quad \frac{h^{n-1}}{(n-1)!}(1-\theta)^{n-1} f^{n}(a + \theta h) - A = 0$$

$$\Rightarrow \quad A = \frac{h^{n-1}}{(n-1)!}(1-\theta)^{n-1} f^{n}(a + \theta h)$$

Putting this value of A in (4), we get

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a)$$

$$+h.\left(\frac{h^{n-1}}{(n-1)!}(1-\theta)^{n-1} f^{n}(a + \theta h)\right)$$

i.e., $$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a)$$

$$+\frac{h^{n}}{(n-1)!}(1-\theta)^{n-1} f^{n}(a + \theta h)$$

Cor. Maclaurin's Theorem with Cauchy's form of remainder.

Putting a = 0, h = x in the above theorem, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots. + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + \frac{x^{n}}{(n-1)!}(1-\theta)^{n-1} f^{n}(\theta x); 0 < \theta < 1 \text{ which}$$

is the Maclaurin's theorem with Cauchy's form of remainder.

Remarks

$R_n = \frac{x^n(1-\theta)^{n-1}}{(n-1)!} f^n(\theta x)$ is called Cauchy's form of remainder after n terms in Maclaurin's expansion of f(x).

Example 1 : Prove that $\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} + ... + (-1)^n \frac{x^{2n}}{\lfloor 2n} + (-1)^{n+1} \frac{x^{2n+1}}{\lfloor 2n+1} \sin(\theta x)$

Solution : $f(x) = \cos x \Rightarrow f(0) = 1$

$f'(x) = -\sin x \Rightarrow f'(0) = 0$

$f''(x) = -\cos x \Rightarrow f''(0) = -1$

$f'''(x) = \sin x \Rightarrow f'''(0) = 0$

…………………………………

$$f^n(x) = \cos\left(x + \frac{n\pi}{2}\right)$$

$$f^{2n+1}(x) = \cos\left(x + (2n+1)\frac{\pi}{2}\right)$$

$$= \cos\left(x + n\pi + \frac{\pi}{2}\right)$$

$$= -\sin(x + n\pi)$$

$$f^{2n+1}(x) = (-1)^{n+1} \sin x$$

$$f^{2n+1}(\theta x) = (-1)^{n+1} \sin(\theta x)$$

$f^{2n}(x) = \cos(x + n\pi)$

$f^{2n}(x) = (-1)^n \cos x$

$f^{2n}(0) = (-1)^n$

By Maclaurin's Series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + ... + \frac{x^{2n}}{\lfloor 2n} f^{2n}(0) + \frac{x^{2n+1}}{\lfloor 2n+1} f^{2n+1}(\theta x)$$

$$\Rightarrow \quad \cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} + ... + (-1)^n \frac{x^{2n}}{\lfloor 2n} + (-1)^{n+1} \frac{x^{2n+1}}{\lfloor 2n+1} \sin(\theta x)$$

Example 2 : Expand $e^{ax} \sin bx$ by Maclaurin's theorem with Cauchy's form of remainder after n terms.

Solution : Let $f(x) = e^{ax} \sin bx$

$\therefore$ $f'(x) = ae^{ax} \sin bx + e^{ax} b \cos bx$

$= e^{ax} (a \sin bx + b \cos bx)$

$f''(x) = ae^{ax}(a \sin bx + b \cos bx) + e^{ax} (ab \cos bx - b^2 \sin bx)$

$= e^{ax}[(a^2 - b^2)\sin bx + 2ab \cos bx]$

$f'''(x) = ae^{ax}[(a^2 - b^2)\sin bx + 2ab \cos bx] + e^{ax}[(a^2 - b^2)\cos bx - 2ab^2 \sin bx]$

…………………………

We know that $f^n(x) = (a^2 + b^2)^{\frac{n}{2}} e^{ax} \sin\left(bx + n \tan^{-1}\frac{b}{a}\right)$

Let x = 0, $\therefore$ $f(0) = 0$, $f'(0) = b$

$f''(0) = 2ab$, $f'''(0) = b(3a^2 - b^2)$

Remarks

$$f^n(\theta x) = (a^2+b^2)^{\frac{n}{2}} e^{a\theta x} \sin\left(b\theta x + n\tan^{-1}\frac{b}{a}\right)$$

Putting these values in Maclaurin's theorem with Cauchy's form of remainder, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + ... + \frac{x^n(1-\theta)^{n-1}}{(n-1)!}f^n(\theta x) \text{ where } 0<\theta<1$$

$$\therefore \quad e^{ax}\sin bx = 0 + x.b + \frac{x^2}{2!}.2ab + \frac{x^3}{3!}b(3a^2-b^2) + ...$$

$$+\frac{x^n}{(n-1)!}(1-\theta)^n(a^2+b^2)^{\frac{n}{2}} e^{a\theta x}\sin\left(b\theta x + n\tan^{-1}\frac{b}{a}\right)$$

$$= bx + 2ab\frac{x^2}{2!} + b(3a^2-b^2)\frac{x^3}{3!} + ...$$

$$+(a^2+b^2)^{\frac{n}{2}}(1-\theta)^n\frac{x^n}{(n-1)!}e^{a\theta x}\sin\left(b\theta x + n\tan^{-1}\frac{b}{a}\right)$$

Exercise 2.1

1. Expand a^x by Maclaurin's theorem with Lagrange's form of remainder after n terms.
2. Show by means of Maclaurin's expansion that

   (i) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - .... + (-1)^{n-2}\frac{x^{n-1}}{(n-1)} + (-1)^{n-1}\frac{x^n}{n(1+\theta)^n}$

   (ii) $e^{ax}\cos bx = 1 + ax + (a^2-b^2)\frac{x^2}{2!} + a(a^2-3b^2)\frac{x^3}{3!} + ...$

   $$... + \frac{x^n}{n!}(a^2+b^2)^{\frac{n}{2}}e^{a\theta x}\cos\left(b\theta x + n\tan^{-1}\frac{b}{a}\right)$$

3. Show that $\sin x + \cos x = 1 + x - \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4(\sin\theta x + \cos\theta x)$
4. Prove that $\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} + ... + (-1)^{n-1}\frac{x^{2n-1}}{\lfloor 2n-1} + (-1)\frac{x^{2n}}{\lfloor 2n}\sin(\theta x),\ 0<\theta<1$
5. Show that $\log(x+h) = \log x + \frac{h}{x} - \frac{h^2}{2x^2} + ... + (-1)^{n-1}\frac{h^n}{n(x+\theta h)^n}.$

2.5 Taylor's Infinite Series

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{\lfloor 2}f''(x) + \frac{h^3}{\lfloor 3}f'''(x) + ...$$

Maclaurin's infinite Series

Put $x = 0$ $h = x$

Remarks

$$f(x) = f(0) + xf'(0) + \frac{x^2}{\lfloor 2}f''(0) + \frac{x^3}{\lfloor 3}f'''(0) + ...$$

Example 1 : Expand e^{2x} in power of x by Maclaurin's series.

Solution : $f(x) = e^{2x} \Rightarrow f(0) = e^0 = 1$

$f'(x) = e^{2x}.2 \Rightarrow f'(0) = 2$

$f''(x) = e^{2x}.4 \Rightarrow f''(0) = 4$

……………………………………

By maclaurin's Series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{\lfloor 2}f''(0) + ...$$

$$= 1 + x(2) + \frac{x^2}{\lfloor 2}(4) + ...$$

$$e^{2x} = 1 + 2x + \frac{4x^2}{\lfloor 2} + ...$$

Example 2 : Expand $\cos^{-1}x$ by Maclaurin's Series.

Solution : $f(x) = \cos^{-1}x \qquad f(0) = \cos^{-1}0 = \frac{\pi}{2}$

$$f'(x) = \frac{-1}{\sqrt{1-x^2}} \qquad f'(0) = -1$$

$$f'(x) = -(1-x^2)^{-\frac{1}{2}}$$

$$= -\left(1 + \frac{1}{2}x^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{\lfloor 2}(-x^2)^2 + ...\right)$$

$$= -\left(1 + \frac{x^2}{2} + \frac{3}{8}x^4 + ...\right)$$

$$f'(x) = -1 - \frac{x^2}{2} - \frac{3}{8}x^4 - ....$$

$$f''(x) = \frac{-2x}{2} - \frac{3}{8}4x^3 - .... \qquad f''(0) = 0$$

$$f'''(x) = -1 - \frac{3}{8} \times 4 \times 3x^2 - .... \qquad f'''(0) = -1$$

and so on.

By Maclaurin's Series $\quad f(x) = f(0) + xf'(0) + \frac{x^2}{\lfloor 2}f''(0) + \frac{x^3}{\lfloor 3}f'''(0) + ...$

$$\Rightarrow \qquad \cos^{-1}x = \frac{\pi}{2} - x + 0 - \frac{x^3}{\lfloor 3}...........$$

Remarks

$$= \frac{\pi}{2} - x - \frac{x^3}{\lfloor 3}...............$$

Example 3 : Expand sin x and cos x in powers of x and hence find cos 18° upto four decimal places.

Solution : (a) Let $f(x) = \sin x \Rightarrow f(0) = 0$

$f'(x) = \cos x \Rightarrow f'(0) = 1$

$f''(x) = -\sin x \Rightarrow f''(0) = 0$

$f'''(x) = -\cos x \Rightarrow f'''(0) = -1$

$f^{iv}(x) = \sin x \Rightarrow f^{iv}(0) = 0$

$f^{v}(x) = \cos x \Rightarrow f^{v}(0) = 1$

……………………………………

By Maclaurin's expansion,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + ...$$

$\therefore$
$$\sin x = 0 + x.1 + \frac{x^2}{2!}.0 + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}.0 + \frac{x^5}{5!}.1 - ....$$
$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...$$

(b) Let $f(x) = \cos x \Rightarrow f(0) = 1$

$f'(x) = -\sin x \Rightarrow f'(0) = 0$

$f''(x) = -\cos x \Rightarrow f''(0) = -1$

$f'''(x) = \sin x \Rightarrow f'''(0) = 0$

$f^{iv}(x) = \cos x \Rightarrow f^{iv}(0) = 1$

$f^{v}(x) = -\sin x \Rightarrow f^{v}(0) = 0$

$f^{vi}(x) = -\cos x \Rightarrow f^{vi}(0) = -1$

$\therefore$
$$\cos x = 1 + x.0 + \frac{x^2}{2!}.(-1) + \frac{x^3}{3!}.0 + \frac{x^4}{4!}.1 + \frac{x^5}{5!}.0 - \frac{x^6}{6!}.(-1) + ...$$
$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$$

Let $x = 18° = \frac{\pi}{10} = 0.314$

$\therefore$
$$\cos 18° = 1 - \frac{1}{2!}\left(\frac{\pi}{10}\right)^2 + \frac{1}{4!}\left(\frac{\pi}{10}\right)^4 - \frac{1}{6!}\left(\frac{\pi}{10}\right)^6 + ...$$
$$= 1 - \frac{1}{2!}(0.314)^2 + \frac{1}{4!}(0.314)^4 - \frac{1}{6!}(0.314)^6 + ...$$

= 1 – 0.04929 + 0.00040 + ….
= 0.95111 (nearly)
= 0.9511 (upto four places of decimal).

Example 4 : Show that $\log(1+\sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + ...$

Solution : $f(x) = \log(1 + \sin x)$

$$= \log\left(1 + x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - .....\right)$$

$$= \log(1+X) \qquad \text{where} \qquad X = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5}......$$

Remarks

$$= X - \frac{X^2}{2} + \frac{X^3}{3} - \frac{X^4}{4} + \dots\dots\dots$$

$$= \left(x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} \dots \right) - \frac{\left(x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} \dots \right)^2}{2} + \frac{\left(x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} \dots \right)^3}{3} - \dots\dots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \dots$$

Example 5 : (i) If $f(x) = x^3 + 2x^2 - 5x + 11$, find the value of $f\left(\frac{9}{10}\right)$ with the help of Taylor's series for $f(x + h)$.

(ii) If $f(x) = x^3 - 2x + 5$, find the value of $f(2.001)$ with the help of Taylor's theorem. Find the approximate change in the value of $f(x)$ when x changes from 2 to 2.001.

Solution : (i) $f(x) = x^3 + 2x^2 - 5x + 11$

Now by Taylor's theorem, we have

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots\dots \quad (1)$$

Putting x = 1 and $h = -\frac{1}{10}$ in (1), we get

$$f\left(1 - \frac{1}{10}\right) = f(1) - \frac{1}{10}f'(1) + \frac{1}{2!}\left(\frac{-1}{10}\right)^2 f''(1) + \frac{1}{3!}\left(-\frac{1}{10}\right)^3 f'''(1) + \dots\dots \quad (2)$$

Here $f(x) = x^3 + 2x^2 - 5x + 11 \Rightarrow f(1) = 9$

$f'(x) = 3x^2 + 4x - 5 \Rightarrow f'(1) = 2$

$f''(x) = 6x + 4 \Rightarrow f''(1) = 10$

$f'''(x) = 6 \Rightarrow f'''(1) = 6$

Putting these values in (2), we have

$$f\left(\frac{9}{10}\right) = 9 - \frac{1}{10}(2) + \frac{1}{200}(10) + \frac{1}{6000}(6)$$

$$= 9 - 0.2 + 0.05 - 0.001 = 8.849$$

(ii) Here in Taylor's series, for x = 2 and h = 0.001, we have

$$f(2.001) = f(2 + .001) = f(2) + (.001)f'(2) + \frac{(.001)^2}{2!}f''(2) + \frac{(.001)}{3!}f'''(2) + \dots \quad (1)$$

Here $f(x) = x^3 - 2x + 5 \Rightarrow f(2) = 9$

$f'(x) = 3x^2 - 2 \Rightarrow f'(2) = 10$

$f''(x) = 6x \Rightarrow f''(2) = 12$

$f'''(x) = 6 \Rightarrow f'''(2) = 6$

Putting these values in (1), we get

$$f(2.001) = 9 + (.001)(10) + \frac{1}{2}(.001)^2(12) + \frac{1}{6}(.001)^3(6) + \dots$$

Remarks

= 9 + .01 + .000006 + .000000001

= 9.010006001 = 9.01 (nearly)

Required approximate change = f(2.001) – f(2) = 9.01 – 9 = .01(nearly)

\

Exercise 2.2

1. Show by means of Maclaurin's theorem that $\log_e(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + ...$
2. Expand by Maclaurin's theorem $\frac{e^x}{e^x+1}$ as far as x^3.
3. Show that $\log(1+\tan x) = x - \frac{x^2}{2} + \frac{2x^3}{3} - \frac{7x^4}{12} + ...$
4. Show that $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x} = \frac{1}{2}\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...\right).$
5. Show that $\tan^{-1}\frac{2x}{1-x^2} = 2\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...\right).$
6. Expand e^{ax} sin bx as an infinite series of ascending powers of x. Give also the (n + 1)th term of the series.
7. Expand a^x by Maclaurin's Series.
8. Expand $\sin(e^x - 1)$ upto and including the term of x^4.
9. Expand $\sin^{-1}x$ in power of x.
10. Expand sin(x+y) in powers of y and deduce that sin(x+y)=sin x cos y+sin y cos x. Hence find sin 31°.
11. (i) If $f(x) = x^3 + 8x^2 + 15x - 24$, calculate the value of $f\left(\frac{11}{10}\right)$ by the application of Taylor's series.

    (ii) Given $f(x) = 3x^3 - 5x^2 + 7$. Calculate $f\left(\frac{21}{10}\right)$ with the help of Taylor's expansion and find the change in the value of f(x) when x change from 2 to 2.1.

    (iii) If $f(x) = x^3 - 6x^2 + 7$, find the value of $f\left(\frac{21}{20}\right)$ by Taylor's theorem.
12. (i) Calculate the approximate value of $\sqrt{26}$ to three decimal places by Taylor's expansion.

    (ii) Calculate the approximate value $\sqrt{17}$ to four decimal places by taking the first three terms of a Taylor's expansion.

Answers

2. $\frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + ...$

6. $bx + abx^2 + \frac{b(3a^2-b^2)x^3}{3!} + ... + \frac{(a^2+b^2)^{\frac{n}{2}}}{n!} x^n \sin\left(n\tan^{-1}\frac{b}{a}\right) + ...$

10. (i) $\sin x + y\cos x - \frac{y^2}{2!}\sin x - \frac{y^3}{3!}\cos x + \frac{y^4}{4!}\sin x + ...$ and sin31° = 0.5150

Remarks

11. (i) 3.511 (ii) 12.733; 1.733 (iii) 1.5426
12. (i) 5.099 (ii) 4.1231

SOLVED EXAMPLE

Example 1 : Expand sin x in powers of $\left(x-\frac{\pi}{4}\right)$.

Solution : $f(x) = \sin x$

$$f(x) = f\left(\frac{\pi}{4}+x-\frac{\pi}{4}\right)$$

$$a = \frac{\pi}{4},\ h = x-\frac{\pi}{4}$$

By Taylor's series

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{\lfloor 2}f''(a) + ...$$

(1)

$$f(x) = \sin x, \qquad f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$f'(x) = \cos x, \qquad f'\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$f''(x) = -\sin x, \qquad f''\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$

$$f'''(x) = -\cos x, \qquad f'''\left(\frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$

Putting in (1)

$$\sin x = f\left(\frac{\pi}{4}\right) + \left(x-\frac{\pi}{4}\right)f'\left(\frac{\pi}{4}\right) + \frac{\left(x-\frac{\pi}{4}\right)^2}{\lfloor 2}f''\left(\frac{\pi}{4}\right) + ...$$

$$= \frac{1}{\sqrt{2}} + \left(x-\frac{\pi}{4}\right)\left(\frac{1}{\sqrt{2}}\right) + \frac{\left(x-\frac{\pi}{4}\right)^2}{\lfloor 2}\left(-\frac{1}{\sqrt{2}}\right) + ...$$

$$= \frac{1}{\sqrt{2}} + \left(x-\frac{\pi}{4}\right)\left(\frac{1}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}}\left(\frac{\left(x-\frac{\pi}{4}\right)^2}{\lfloor 2}\right) - ......$$

Example 2 : If $0 < x \le 2$, then prove that :

$$\log x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + ...$$

Solution : Let $f(x) = \log x$

$\therefore$ $\log x = f(x) = f[1 + (x-1)]$

$= f(1+h)$ where $h = (x-1)$

Remarks

$$= f(1) + hf'(1) + \frac{h^2}{2!}f''(1) + \frac{h^3}{3!}f'''(1) + \frac{h^4}{4!}f^{iv}(1) + \ldots.$$

(1)

Here $f(x) = \log x \Rightarrow f(1) = 0$

$$f'(x) = \frac{1}{x} \Rightarrow f'(1) = 1$$

$$f''(x) = -\frac{1}{x^2} \Rightarrow f''(1) = -1$$

$$f'''(x) = \frac{2}{x^3} \Rightarrow f'''(1) = 2$$

$$f^{iv}(x) = -\frac{6}{x^4} \Rightarrow f^{iv}(1) = -6$$

…………………………………..

Putting these values in (1), we have

$$\log x = h - \frac{h^2}{2!} + \frac{h^3}{3!}.(2) + \frac{h^4}{4!}.(-6) + \ldots..$$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots..$$

Example 3 : Prove that $f(ax) = f(x) + (a-1)xf'(x) + \frac{(a-1)^2}{\underline{|2}}x^2 f''(x) + \ldots..$

Solution : $f(ax) = f(x + (a-1)x)$

$a = x, \quad h = (a-1)x$

By Taylor's series

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{\underline{|2}}f''(a) + \ldots..$$

$$f(x + (a-1)x) = f(x) + (a-1)xf'(x) + \frac{(a-1)^2 x^2}{\underline{|2}}f''(x) + \ldots..$$

$$f(ax) = f(x) + x(a-1)f'(x) + \frac{x^2}{\underline{|2}}(a-1)^2 f''(x) + \ldots..$$

Exercise 2.3

1. Expand a^x in powers of $(x - a)$.
2. Expand $4x^2 + 7x + 5$ in powers of $(x + 2)$.
3. Expand sin x in ascending powers of $\left(x - \frac{\pi}{2}\right)$.
4. Prove that $\frac{1}{x} = \frac{1}{a} - \frac{x-a}{a^2} + \frac{(x-a)^2}{a^3} - \ldots..$

Remarks

5. Prove that $f\left(\frac{x^2}{1+x}\right) = f(x) + \frac{x}{1+x} f'(x) + \left(\frac{x}{1+x}\right)^2 \frac{1}{\underline{|2}} f''(x) + \ldots\ldots$

Answers

1. $a^a\left[1+(x-a)\log a + \frac{(x-a)^2}{2!}(\log a)^2 + \frac{(x-a)^3}{3!}(\log a)^3 + \ldots\right]$
2. $7 - 9(x+2) + 4(x+2)^2 + \ldots.$
3. $1 - \frac{\left(x-\frac{\pi}{2}\right)^2}{2!} + \frac{\left(x-\frac{\pi}{2}\right)^4}{4!} - \ldots.$

2.6 EXPANSION BY DIFFERENTIAL EQUATIONS

Example 1 : Show that

$$e^{a\sin^{-1}x} = 1 + ax + \frac{a^2x^2}{2!} + \frac{a(a^2+1)}{3!}x^3 + \frac{a^2(a^2+2^2)}{4!}x^4 + \frac{a(a^2+1)(a^2+3^2)}{5!}x^5 + \ldots$$

and deduce that $e^{\theta} = 1 + \sin\theta + \frac{1}{2!}\sin^2\theta + \frac{2}{3!}\sin^3\theta + \ldots.$

Solution : Let $y = e^{a\sin^{-1}x}$ (1)

$\therefore$ $y_1 = e^{a\sin^{-1}x}.\frac{a}{\sqrt{1-x^2}}$ (2)

$\therefore$ $\sqrt{1-x^2}\,y_1 = ay$

Squaring both sides, we have

$(1-x^2)y_1^2 = a^2y^2$

Differentiating again, we have

$(1-x^2)2y_1y_2 - 2xy_1^2 = a^2 2yy_1$

$(1-x^2)y_2 - xy_1 = a^2y$ (3)

Differentiating both sides w.r.t. x, n times by Leibnitz's theorem,

$$y_{n+2}(1-x^2) + {}^nC_1y_{n+1}(-2x) + {}^nC_2y_n(-2) - [y_{n+1}x + {}^nC_1y_n.1] = a^2y_n$$

$\therefore$ $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$

Let $x = 0$

$\therefore$ $y_{n+2}(0) = (n^2+a^2)y_n(0)$ (A)

From (1), $y(0) = 1$

From (2), $y_1(0) = a$

From (3), $y_2(0) = a^2$

Putting n = 1, 2, 3, in (A), we get

$y_3(0) = (1^2+a^2)y_1(0) = (a^2+1^2)a$

$y_4(0) = (a^2+2^2)y_2(0) = (a^2+2^2)a^2$

$y_5(0) = (a^2+3^2)y_3(0) = (a^2+3^2)(a^2+1^2)a$

By Maclaurin's theorem, we have

Remarks

$$y = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + ...$$

$$e^{a\sin^{-1}x} = 1 + ax + \frac{a^2x^2}{2!} + \frac{(a^2+1^2)}{3!}x^3 + \frac{(a^2+2^2)a^2}{4!}x^4 + ...$$

Putting a = 1, x = sin θ, we get

$$e^{\theta} = 1 + \sin\theta + \frac{1}{2!}\sin^2\theta + \frac{2}{3!}\sin^3\theta + \frac{5}{4!}\sin^4\theta + ...$$

Example 2 : Prove that

$$(x + \sqrt{1+x^2})^n = 1 + nx + \frac{n^2x^2}{2!} + \frac{n(n^2-1^2)}{3!}x^3 + \frac{n^2(n^2-2^2)}{4!}x^4 + ...$$

Solution : Let $y = (x + \sqrt{1+x^2})^n$

(1)

$\therefore$ $$y_1 = n(x + \sqrt{1+x^2})^{n-1}\left[1 + \frac{2x}{2\sqrt{1+x^2}}\right]$$

$$= n(x + \sqrt{1+x^2})^{n-1}\frac{(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}$$

$$= n\frac{(x+\sqrt{1+x^2})^n}{\sqrt{1+x^2}}$$

$\therefore$ $$\sqrt{1+x^2}\,y_1 = ny$$

Squaring, $$(1+x^2)y_1^{\,2} = n^2y^2$$

(2)

Differentiating, $(1 + x^2)2y_1y_2 + 2xy_1^{\,2} = n^2 2yy_1$

or $$(1 + x^2)y_2 + xy_1 = n^2y$$

(3)

Differentiating both sides w.r.t. x, m times, we get

$$y_{m+2}(1+x^2) + {}^mC_1y_{m+1}2x + {}^mC_2y_{m}.2 + y_{m+1}x + {}^mC_1y_m.1 = n^2y_m$$

or $$(1 + x^2)y_{m+2} + (2m + 1)xy_m + 1 + (m^2 - n^2)y_m = 0$$

when x = 0

$$y_{m+2}(0) = (n^2 - m^2)y_m(0)$$

(4)

From (1), $y(0) = 1$

From (2), $y_1(0) = n$

From (3), $y_2(0) = n^2$

From (4), $y_3(0) = (n^2 - 1^2)y_1(0) = (n^2 - 1^2)n$

$y_4(0) = (n^2 - 2^2)y_2(0) = (n^2 - 2^2)n^2$

$y_4(0) = (n^2 - 3^2)y_3(0) = (n^2 - 3^2)(n^2 - 1^2)n$

By Maclaurin's theorem, we get

$$y = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + ...$$

Remarks

or $$y = 1 + nx + \frac{n^2x^2}{2!} + \frac{n(n^2-1^2)}{3!}x^3 + \frac{n^2(n^2-2^2)}{4!}x^4 + \frac{n(n^2-3^2)(n^2-1^2)}{5!}x^5 + \dots$$

Exercise 2.4

1. Prove that $\sin(m\sin^{-1}x) = mx + \frac{m(1^2-m^2)}{3!}x^3 + \frac{m(1^2-m^2)(3^2-m^2)}{5!}x^5 + \dots$
2. If $y = (\sin^{-1}x)^2$, show that
   (i) $(1 - x^2)y_2 - xy_1 = 2$.
   (ii) $\frac{(\sin^{-1}x)^2}{2} = \frac{x^2}{2!} + \frac{2^2}{4!}x^4 + \frac{2^2.4^2}{6!}x^6 + \dots..$
3. Expand $\sin^{-1}\frac{2x}{1+x^2} = 2\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \dots.\right)$.
4. Expand $\tan^{-1}\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}} = \frac{x^2}{2} + \frac{x^6}{12} + \dots.$

2.7 ASYMPTOTES

Definition 1 : A straight line, at a finite distance from the origin is said to be an asymptote to an infinite branch of a curve, if the perpendicular distance of a point P on that branch from the straight line tends to zero, as $P \to \infty$ along the branch.

Definition 2 : An asymptote to an infinite branch of a curve is the limiting position of the tangent whose point of contact tends to infinity along the branch, but which itself remains at a finite distance from the origin.

Working Rule to find Asymptotes

1. Asymptote | | x-axis :

Putting the co-efficient of highest degree of x = 0.

2. Asymptote | | y-axis :

Putting the co-efficient of highest power of y = 0.

3. Oblique asymptote :

Putting x = 1, y = m

Then we will find

$\phi_n(m)$ = (Total degree of x & y = n)

$\phi_{n-1}(m)$ = (Total degree of x & y = n – 1)

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…………………………………………..

$\phi_1(m)$

$\phi_0(m)$

Putting $\phi_n(m) = 0$

$\Rightarrow$ we will find values of m

Case I : If all values of m are distinct

$m = m_1, m_2, m_3, \dots..$

Then

$$C = \frac{-\phi_{n-1}(m)}{\phi'_n(m)}$$

$y = mx + C$ is the asymptote.

Remarks

Case II : If two values of m are same

say $m = m_1, m_1$

Then we will evaluate C from the equation

$$\frac{C^2}{\underline{|2}}\phi_n''(m) + \frac{C}{\underline{|1}}\phi_{n-1}'(m) + \phi_{n-2}(m) = 0$$

$y = mn + C$ is the asymptote.

Case III : If three values of m are same

$m = m_1, m_1, m_1$

Then evaluate C from the equation

$$\frac{C^3}{\underline{|3}}\phi_n'''(m) + \frac{C^2}{\underline{|2}}\phi_{n-1}''(m) + \frac{C}{\underline{|1}}\phi_{n-2}'(m) + \phi_{n-3}(m) = 0$$

Then $y = mn + C$ is the asymptote.

Example 1 : Find all the asymptotes of the following curves :

(a) $x^3 + 3x^2y - 4y^3 - x + y + 3 = 0$.

(b) $2x^3 - x^2y - 2xy^2 + y^3 + 2x^2 - 7xy + 3y^2 + 2x + 2y + 1 = 0$.

Solution : (a) The given curve is $x^3 + 3x^2y - 4y^3 - x + y + 3 = 0$ 91)

Asymptotes parallel to the co-ordinate axes :

Since the co-efficients of highest degree terms of x and y, i.e. x^3 and y^3 are constants, therefore the curve has no asymptotes parallel to x-axis or y-axis.

Oblique Asymptotes :

Putting x = 1, m = m in the third, second and first degree terms in (1) one by one, we get

$$\phi_3(m) = 1 + 3m - 4m^3,$$
$$\phi_2(m) = 0$$
$$\phi_1(m) = -1 + m$$

Now $\phi_3(m) = 0 \Rightarrow 1 + 3m - 4m^3 = 0$

$\Rightarrow (1 - m)(1 + 4m + 4m^2) = 0$

$\Rightarrow (1 - m)(1 + 2m)^2 = 0$

$\Rightarrow m = 1, -\frac{1}{2}, -\frac{1}{2}$

Also $\phi_3'(m) = 3 - 12m^2$ and $\phi_3''(m) = -24m$

$$\therefore \quad c = -\frac{\phi_{n-1}(m)}{\phi_n'(m)} = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{0}{3 - 12m^2}$$

When m = 1, $c = -\frac{0}{3-12} = 0$

When $m = -\frac{1}{2}$, $c = -\frac{0}{3-3} = \frac{0}{0}$ [Indeterminate form]

$\therefore$ In this case, c is given by

$$\frac{c^2}{2!}\phi_3''(m) + c\phi_2'(m) + \phi_1(m) = 0$$

or $$\frac{c^2}{2!}(-24m) + c.0 + (m-1) = 0$$

or $$\frac{c^2}{2}.(12) - \frac{1}{2} - 1 = 0 \qquad \left[\because m = -\frac{1}{2}\right]$$

Remarks

$$c^2 = \frac{1}{4} \Rightarrow c = \pm\frac{1}{2},\ \text{for } m = -\frac{1}{2}$$

Now the asymptotes are obtained by putting the values of m and c in $y = mx + c$,

$$y = x + 0,\ y = -\frac{1}{2}x + \frac{1}{2} \quad \text{and} \quad y = -\frac{1}{2}x - \frac{1}{2}$$

or $x - y = 0,\ x + 2y - 1 = 0$ and $x + 2y + 1 = 0$

which are the two required asymptotes.

(b) The given curve is

$$2x^3 - x^2y - 2xy^2 + y^3 + 2x^2 - 7xy + 3y^2 + 2x + 2y + 1 = 0 \quad (1)$$

Since the co-efficients of x^3 and y^3 are constants, therefore there are no asymptotes parallel to the co-ordinate axes.

Oblique Asymptotes :

Putting $x = 1$, $y = m$ in the third, second and first degree terms, one by one in (1), we get

$$\phi_3(m) = 2 - m - 2m^2 + m^3$$
$$\phi_2(m) = 2 - 7m + 3m^2$$
$$\phi_1(m) = 2 + 2m$$

Now $\phi'_3(m) = -1 - 4m + 3m^2$

Slopes of asymptotes are roots of equation $\phi_3(m) = 0$

i.e., $m^2(m - 2) - (m - 2) = 0$

or $(m - 2)(m^2 - 1) = 0$, which gives $m = 2, 1, -1$

Now c is given by $c = -\dfrac{\phi_2(m)}{\phi'_3(m)} = -\dfrac{3m^2 - 7m + 2}{3m^2 - 4m - 1}$

$\therefore$ For $m = 2$, $c = -\dfrac{12 - 14 + 2}{12 - 8 - 1} = 0$

For $m = 1$, $c = -\dfrac{3 - 7 + 2}{3 - 4 - 1} = -\dfrac{2}{2} = -1$

For $m = -1$, $c = -\dfrac{3 + 7 + 2}{3 + 4 - 1} = -2$

Thus the asymptotes are $y = 2x$, $y = x - 1$ and $y = -x - 2$.

Example 2 : Find all the asymptotes to the curve

$$(x - y)^2 (x - 2y)(x - 3y) - 2a(x^3 - y^3) - 2a^2(x - 2y)(x + y) = 0.$$

Solution : The equation of the curve is

$$(x - y)^2 (x - 2y)(x - 3y) - 2a(x^3 - y^3) - 2a^2(x - 2y)(x + y) = 0 \quad (1)$$

Asymptotes parallel to the co-ordinate axes :

Since the coefficients of x^4 and y^4 are constant, therefore there are no asymptotes parallel to the co-ordinate axes.

Oblique Asymptotes :

Putting $x = 1$, $y = m$ in the fourth, third and second degree terms of (1), one by one, we get

$$\phi_4(m) = (1 - m)^2(1 - 2m)(1 - 3m)$$
$$\phi_3(m) = -2a(1 - m^3)$$
$$\phi_2(m) = -2a^2(1 - 2m)(1 + m)$$

Slopes of the asymptotes are roots of equation $\phi_3(m) = 0$

i.e., $(1 - m)^2(1 - 2m)(1 - 3m) = 0 \quad \Rightarrow \quad m = 1, 1, \dfrac{1}{2}, \dfrac{1}{3}$

Remarks

Now $\phi_4'(m) - 2(1-m)(1-2m)(1-3m) - 2(1-m)^2(1-3m) - 3(1-m)^2(1-2m)$

$$\phi_4''(m) = -2[-(1-2m)(1-3m) - 2(1-m)(1-3m) - 3(1-m)(1-2m)]$$
$$-2[-2(1-m)(1-3m) - 3(1-m)^2] - 3[-2(1-2m) - 2(1-m)]$$

and $\phi_3'(m) = 6am^2$

When $m = 1$, $\phi_4''(m) = 4$, $\phi_3'(m) = 6a$, $\phi_2 = 4a^2$

Thus c is given by $\frac{c^2}{2!}\phi_3''(m) + \frac{c}{1!}\phi_2'(m) + \phi_1(m) = 0$

i.e., $\frac{c^2}{2}.4 + c.6a + 4a^2 = 0$

or $c^2 + 3ac + 2a^2 = 0$

or $(c+a)(c+2a) = 0 \quad \Rightarrow \quad c = -a, -2a.$

Thus the two corresponding parallel asymptotes are $y = x - a$ and $y = x - 2a$.

When $m = \frac{1}{2}$ $\quad \phi_4'(m) = -2\left(\frac{1}{4}\right)\left(-\frac{1}{2}\right) = \frac{1}{4},$

$$\phi_3(m) = -2a\left(1 - \frac{1}{8}\right) = -\frac{7a}{4}$$

$\therefore$ $$c = -\frac{\phi_3(m)}{\phi_4'(m)} = \frac{\frac{7a}{4}}{\frac{1}{4}} = 7.a$$

$\therefore$ The corresponding asymptote is

$$y = \frac{1}{2}x + 7a \quad \Rightarrow \quad x - 2y + 14a = 0$$

When m $\frac{1}{3}$, $\quad \phi_4'(m) = -3\left(\frac{4}{9}\right)\left(\frac{1}{3}\right) = -\frac{4}{9}$

$$\phi_3(m) = -2a\left(\frac{26}{27}\right) = -\frac{52}{27}a$$

$\therefore$ $$c = -\frac{\phi_3(m)}{\phi_4'(m)} = -\frac{-\frac{52}{27}a}{-\frac{4}{9}} = -\frac{52}{27}a \times \frac{9}{4} = -\frac{13}{3}a$$

Thus the corresponding asymptote is

$$y = \frac{1}{3}x - \frac{13}{3}a \quad \Rightarrow \quad x - 3y - 13a = 0$$

Hence the asymptote to the curve are

$$y = x - a,\ y = x - 2a,\ x - 2y + 14a = 0,\ x - 3y - 13a = 0.$$

Example 3 : Show that the curve $y^2 - 4ax = 0$ has no asymptotes.

Solution : The given equation of the curve is $y^2 - 4ax = 0$

Asymptotes parallel to the co-ordinate axes :

Remarks

Since the co-efficient of y^2, i.e. highest degree term in y is constant, therefore there is no asymptote parallel to y-axis. Also as the co-efficient of x, the highest degree term in x is constant, so there is no asymptote parallel to x-axis.

Oblique Asymptotes :

Putting x = 1, y = m in the second and first degree terms in the given equation, we get

$$\phi_2(m) = m^2, \qquad \phi_1(m) = -4a$$

Slopes of asymptotes are roots of equation $\phi_3(m) = 0$

i.e., $m^2 = 0 \quad \Rightarrow \quad m = 0, 0.$

But the asymptotes (if any) for m = 0, correspond to asymptotes parallel to the x-axis. But we have seen above that there is no asymptote parallel to x-axis. Thus the curve has no oblique asymptotes.

Therefore the given curve has no asymptotes.

Example 4 : Find the asymptotes of the curve $x(x^2 - y^2) + y(3y - x) = 0$ and prove that three points of intersection of the curve and its asymptotes lie on $7x - 3y + 6 = 0$.

Solution : The given equation is

$$x(x^2 - y^2) + y(3y - x) = 0$$

or $x^3 - xy + 3y^2 = 0$ (1)

Asymptotes parallel to co-ordinate axes :

The co-efficient of x^3 is constant and so there is no asymptote parallel to x-axis.

The co-efficient of y^2 is $-x + 3$ and so asymptote parallel to y-axis is $x = 3$.

Oblique Asymptotes :

Putting x = 1 and y = m in third, second and first degree terms of (1), we have

$$\phi_3(m) = 1 - m^2 \qquad \phi_3'(m) = -2m$$

$$\phi_2(m) = 3m^2 - m$$

Slopes of asymptotes are roots of equation $\phi_3(m) = 0$

i.e., $1 - m^2 = 0 \quad \Rightarrow \quad m = \pm 1$

For m = 1, $$c = \frac{-\phi_2(m)}{\phi_3'(m)} = \frac{-(3m^2 - m)}{-2m} = \frac{3-1}{2} = 1$$

The corresponding asymptote is y = mx + c

i.e., $y = x + 1$ or $x - y + 1 = 0$

For m = – 1 $$c = \frac{-\phi_2(m)}{\phi_3'(m)} = \frac{3m^2 - m}{2m} = \frac{3+1}{-2} = -2$$

The corresponding asymptote is y = mx + c

i.e., $y = -x - 2$ or $x + y + 2 = 0$

Thus all the asymptotes are

$$x - 3 = 0, \; x - y + 1 = 0 \; \text{ and } \; x + y + 2 = 0$$

The joint equation of asymptotes is

$$(x - 3)(x + y + 2)(x - y + 1) = 0$$

or $x^3 - xy^2 + 3y^2 - xy - 7x + 3y - 6 = 0$ (2)

Subtracting (2) from (1), we get

$$7x - 3y + 6 = 0$$

The number of points of intersections of asymptotes and curve = n(n – 2)=3(3 – 2) = 3

Hence the three points of intersections of the curve and its asymptote lie on $7x - 3y + 6 = 0$.

Alternative method for finding oblique asymptotes

The given equation is $x(x - y)(x + y) = y(x - 3y)$

Remarks

One oblique asymptote is

$$x + y = \lim_{x\to\infty} \frac{y(x-3y)}{x(x-y)} = \lim_{x\to\infty} \frac{\frac{y}{x}\left(1-\frac{3y}{x}\right)}{1-\frac{y}{x}}$$

$$= \frac{-1(1+3)}{1+1} = -2 \qquad \left[\because \lim_{x\to\infty} \frac{y}{x} = -1\right]$$

The other oblique asymptote is

$$x - y = \lim_{x\to\infty} \frac{y(x-3y)}{x(x+y)} = \lim_{x\to\infty} \frac{\frac{y}{x}\left(1-\frac{3y}{x}\right)}{1+\frac{y}{x}} = \frac{1(1-3)}{1+1} = -1 \qquad \left[\because \lim_{x\to\infty} \frac{y}{x} = 1\right]$$

$\therefore$ Oblique asymptotes are $x + y + 2 = 0$, $\quad x - y + 1 = 0$.

Example 5 : Find all the asymptotes of the curve

$$x^2y - xy^2 + xy + y^2 + x - y = 0.$$

Solution : The given equation is

$$x^2y - xy^2 + xy + y^2 + x - y = 0 \qquad (1)$$

Asymptotes parallel to the co-ordinate axes :

The co-efficient of x^2 is y and so asymptote parallel to x-axis is $y = 0$.

The co-efficient of y^2 is $1 - x$ and so asymptote parallel to y-axis is $x = 1$.

Oblique Asymptotes :

Putting $x = 1$ and $y = m$ in the third, second and first degree terms of (1), one by one, we get

$$\phi_3(m) = m - m^2 \qquad \phi_3'(m) = 1 - 2m$$

$$\phi_2(m) = m + m^2$$

$$\phi_1(m) = 1 - m$$

Slopes of the asymptotes are roots of $\phi_3(m) = 0$

i.e., $m - m^2 = 0 \Rightarrow m(1 - m) = 0 \Rightarrow m = 0, 1$

For $m = 0$, the asymptote is parallel to x-axis

For $m = 1$, $\quad c = \frac{-\phi_2(m)}{\phi_3'(m)} = \frac{-(m+m^2)}{1-2m} = \frac{-(1+1)}{1-2} = 2$

$\therefore$ The equation of asymptote is $y = mx + c$ i.e., $y = x + 2$.

Hence all the three asymptotes are $y = 0$, $x = 1$, $y = x + 2$.

Exercise 2.5

Find all the asymptotes of the following curves :

1. $x^3 + 4x^2y + 5xy^2 + 2y^3 + 2x^2 + 4xy + 2y^2 - x - 9y + 1 = 0$.
2. (a) $x^3 + 3xy^2 - x^2y - 3y^3 + x^2 - 2xy + 3y^2 + 4x + 7 = 0$

   (b) $x^3 - 6x^2y + 11xy^2 - 6y^3 + x + y + 1 = 0$.
3. (a) $x^3 + 2x^2y - xy^2 - 2y^3 + 3xy + 3y^2 + x + 1 = 0$.

   (b) $2x^3 - x^2y - 2xy^2 + y^3 - 4x^2 + 8xy + 4x + 1 = 0$.

   (c) $y^3 - x^2y - 2xy^2 + 2x^3 - 7xy + 3y^2 + 2x^2 + 2x + 2y + 1 = 0$.

Remarks

4. (a) $4x^3 - 3xy^2 - y^3 + 2x^2 - xy - y^2 - 1 = 0$.
   (b) $y^3 + x^2y - 2xy^2 - y + 1 = 0$.
   (c) $x^3 + 2x^2y + xy^2 - x^2 - xy + 2 = 0$.
5. $x^3 + y^3 - 3axy = 0$
6. $(x + y)^2(x + y + 2) - x - 9y + 2 = 0$.

Answers

1. $x + 2y + 2 = 0,\ x + y = \pm 2\sqrt{2}$
2. (a) $2x - 2y + 1 = 0$ (b) $x - y = 0,\ x - 2y = 0,\ x - 3y = 0$
3. (a) $x + y = 0,\ x - y + 1 = 0,\ x + 2y = 1$ (b) $x - y + 2 = 0,\ x + y = 2,\ 2x - y = 4$
   (c) $x + y + 2 = 0,\ x - y - 1 = 0,\ y = 2x$
4. (a) $2x + y = 0,\ 2x + y + 1 = 0,\ x - y = 0$ (b) $y = 0,\ x - y = \pm 1$
   (c) $x + y = 0,\ x + y = 1,\ x = 0$
5. $x + y + a = 0$.
6. $y = -x + 2,\ y = -x - 2 + \sqrt{5},\ y = -x - 2 - \sqrt{5}$.

2.8 INTERSECTION OF THE CURVE AND ITS ASYMPTOTES

To prove that any asymptote of an algebraic curve of the nth degree cuts the curve in (n – 2) points.

Proof : Let the curve be

$$x^n\phi_n\left(\frac{y}{x}\right) + x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right) + x^{n-2}\phi_{n-2}\left(\frac{y}{x}\right) + \ldots.. = 0 \qquad (1)$$

and let $y = mx + c$ (2)

be its asymptote. For the points of intersection of (1) and (2), we solve them simultaneously for x and y. Eliminating y, we get

$$x^n\phi_n\left(m + \frac{c}{x}\right) + x^{n-1}\phi_{n-1}\left(m + \frac{c}{x}\right) + x^{n-2}\phi_{n-2}\left(m + \frac{c}{x}\right) + \ldots.. = 0$$

Expanding by Taylor's theorem, we have

$$x^n\left[\phi_n(m) + \frac{c}{x}\phi'_n(m) + \frac{1}{2!}\frac{c^2}{x^2}\phi''_n(m) + \ldots\right] + x^{n-1}\left[\phi_{n-1}(m) + \frac{c}{x}\phi'_{n-1}(m) + \ldots\right]$$

$$+x^{n-2}[\phi_{n-2}(m) + \ldots] + \ldots. = 0$$

Arranging the terms in descending powers of x, we get

$$\phi_n(m)x^n + \{c\phi'_n(m) + \phi_{n-1}(m)\}x^{n-1} + \left\{\frac{c^2}{2!}\phi''_n(m) + c\phi'_{n-1}(m) + \phi_{n-2}(m)\right\}x^{n-2} + \ldots = 0 \qquad (3)$$

But for the asymptotes (i) $\phi_n(m) = 0$ and (ii) $c\phi'_n(m) + \phi_{n-1}(m) = 0$

Putting these values in (3), we have

$$x^{n-2}\left[\frac{c^2}{2!}\phi''_n(m) + c\phi'_{n-1}(m) + \phi_{n-2}(m)\right] + x^{n-3}[\ldots.] = 0$$

which is an equation of (n – 2)th degree and determines (n – 2) values of x. Thus (2) cuts (1) in (n – 2) points.

Hence an asymptote meets its curve in n – 2 points. Because there are in general n asymptotes of a curve of nth degree, therefore the points of intersection of a curve and its asymptotes are n(n – 2).

Remark : Let the equation of the curve with usual notations be written as $F_n+F_{n-2} = 0$, where $F_n = 0$ is the joint equation of its n asymptotes [Cor. (iii) of Art 4.6]. Then points of intersection of the curve and its asymptotes lie on

$$F_n + F_{n-2} + \lambda F_n = 0 \qquad \text{for all real values of } \lambda.$$

Putting $\lambda = -1$, we get F_{n-2}, the curve, on which lies the common points of the curve and its asymptotes and these common points are n(n – 2) in number.

Some Important Examples :

Example 1 : Find the equation of the hyperbola having $x + y - 1 = 0$ and $x - y + 2=0$ as its asymptotes and passing through the origin.

Solution : The equation of hyperbola is a second degree equation in x and y.

$\therefore$ Its equation can be put in the form $F_2 + F_0 = 0$, where

$F_2 = (x + y - 1)(x - y + 2) = 0$ is the joint equation of the two asymptotes and F_0 = some constant say c.

Thus the hyperbola whose asymptotes are given as above is

$$(x + y - 1)(x - y + 2) + c = 0$$

As it passes through the origin (0, 0), therefore putting x = 0, y = 0, we get

$$-2 + c = 0 \qquad \Rightarrow \qquad c = 2$$

Hence the required hyperbola is $(x + y - 1)(x - y + 2) + 2 = 0$.

Example 2 : Show that the four asymptotes of the curve

$$xy(x^2 - y^2) + 25y^2 + 9x^2 - 144 = 0$$

cut it again in eight points on an ellipse whose eccentricity is $\frac{4}{5}$.

Solution : The given equation is of the form $F_4 + F_2 = 0$

Thus each non-repeated factor of F_4 equated to zero gives its asymptotes and the joint equation of asymptotes is

$$F_4 = xy(x - y)(x + y) = 0$$

The four asymptotes are x = 0, y = 0, x – y = 0 and x + y = 0

The four asymptotes will cut the curve again in 4(4 – 2) = 8 points and these points lie on the curve

$$F_2 = 25y^2 + 9x^2 - 144 = 0$$

or $$9x^2 + 25y^2 = 144$$

or $$\frac{x^2}{\left(\frac{144}{9}\right)} + \frac{y^2}{\left(\frac{144}{25}\right)} = 1$$

which is an ellipse.

Remarks

Here $a^2 = \dfrac{144}{9}$ and $b^2 = \dfrac{144}{25}$

The eccentricity of the ellipse is given by

$$b^2 = a^2(1 - e^2)$$

or $$\frac{144}{25} = \frac{144}{9}(1 - e^2)$$

or $$9 = 25(1 - e^2)$$

or $$e^2 = \frac{16}{25} \quad \Rightarrow \quad e = \frac{4}{5}$$

which proves that the eight points of intersection of the curve and its asymptote lie on an ellipse with eccentricity $\dfrac{4}{5}$.

Example 3 : Find the cubic which has the same asymptotes as the curve

$$x^3 - 6x^2y + 11xy^2 - 6y^3 + x + y + 1 = 0$$

and which touches the axis of y at the origin and passes through the point (3, 2).

Solution : The given curve is of the form is $F_3 + F_1 = 0$

Thus each non-repeated factor of F_3 equated to zero gives its asymptote and the joint equation of its asymptotes is

$$F_3 = x^3 - 6x^2y + 11xy^2 - 6y^3 = 0$$

Let the required cubic having the same asymptotes as the given curve be

$$x^3 - 6x^2y + 11xy^2 - 6y^3 + ax + by + c = 0$$

(1)

It passes through the origin and the point (3, 2)

$\therefore$ $$c = 0$$

and $$27 - 108 + 132 - 48 + 3a + 2b = 0 \quad \Rightarrow \quad 3a + 2b + 3 = 0 \qquad (2)$$

Equating the lowest degree terms of (1) to zero, we get the tangent at origin as $ax + by = 0$

But y-axis whose equation is $x = 0$ is a tangent at origin.

Thus $ax + by = 0$ and $x = 0$ represent the same line

$\therefore$ $$b = 0$$

Then from (2), we have $a = -1$. Thus the required cubic is

$$x^3 - 6x^2y + 11xy^2 - 6y^3 - x = 0.$$

Exercise 2.6

1. Show that the points of intersection of the curve

$$4x^3 - 2x^2y - 4xy^2 + 2y^3 + 6y^2 - 14xy + 4x^2 + 6y + 1 = 0$$

and its asymptote lie on $8x + 2y + 1 = 0$.

2. Show that the three points of intersection of the curve

$$2y^3 - 2x^2y - 4xy^2 + 4x^3 - 14xy + 6y^2 + 4x^2 + 6y + 1 = 0$$

and its asymptotes lie on the straight line $8x + 2y + 1 = 0$.

3. Find the equation of the cubic which has the same asymptotes as the curve

$$x^3 - 6x^2y + 11xy^2 - 6y^3 + x + y + 1 = 0$$

and which passes through the points (0, 0), (1, 0) and (0, 1).

4. Find the equation of the cubic which has the same asymptotes as the curve

$$x^3 - 6x^2y - 11xy^2 - 6y^3 + x + y + 4 = 0$$

and which passes through (0, 0), (2, 0) and (0, 2).

5. Find the equation of the quartic curve which has $x = 0$, $y = 0$, $y = x$ and $y = -x$ as its four asymptotes and which passes through (a, b) and whose eight points of intersection with its asymptotes lie on the circle $x^2 + y^2 = a^2$.

   [**Hint.** The equation of the quartic curve is $xy(y^2 - x^2) + k(x^2 + y^2 - a^2) = 0$, where k is constant.]

Answers

3. $x^3 - 6x^2y + 11xy^2 - 6y^3 - x + 6y = 0$
4. $x^3 - 6x^2y + 11xy^2 - 6y^3 - 4x + 24y = 0$
5. $bxy(y^2 - x^2) + a(a^2 - b^2)(x^2 + y^2 - a^2) = 0$

2.9 WORKING RULE FOR FINDING ASYMPTOTES OF POLAR CURVES

1. In the given equation, put $r = \dfrac{1}{u}$.
2. If the given equation involves trigonometric ratios, convert them to sin θ. Find the limit of θ as $u \to \theta$. Let $\theta_1, \theta_2, \ldots, \theta_n$ be the limits obtained.
3. Find $\lim\limits_{\substack{\theta \to \theta_i \\ u \to 0}} \left(-\dfrac{d\theta}{du}\right)$ for each of the θ_I, obtained in step 2.
4. The corresponding asymptotes are given by the equation $p = r \sin(\theta_I - \theta)$.

Note : The following results from Trigonometry which the students have already learnt shall be useful for solving the problems :

1. If $\sin\theta = 0$, then $\theta = n\pi$ where n is any integer.
2. If $\cos\theta = 0$, then $\theta = (2n + 1)\dfrac{\pi}{2}$.
3. $\sin(n\pi + \theta) = (-1)^n \sin\theta$.
4. $\cos(n\pi + \theta) = (-1)^n \cos\theta$.
5. If $\cos\theta = \cos\alpha$, then $\theta = 2n\pi \pm \alpha$.
6. If $\sin\theta = \sin\alpha$, then $\theta = n\pi + (-1)^n\alpha$.
7. If $\tan\theta = \tan\alpha$, then $\theta = n\pi + \alpha$.

Example 1 : Find the asymptotes of the curve $r = a\sec\theta + b\tan\theta$.

Solution : The given curve is

Remarks

$$r = \frac{a}{\cos\theta} + \frac{b\sin\theta}{\cos\theta} = \frac{a + b\sin\theta}{\cos\theta}$$

or $$u = \frac{\cos\theta}{a + b\sin\theta}, \text{where } u = \frac{1}{r} \qquad (1)$$

$\therefore$ $$\frac{du}{d\theta} = \frac{(a + b\sin\theta)(-\sin\theta) - \cos\theta . b\cos\theta}{(a + b\sin\theta)^2}$$

$$= \frac{-[a\sin\theta + b]}{(a + b\sin\theta)^2} \qquad (2)$$

From (1), $u \to 0$ gives $\cos\theta \to 0$

i.e., $\theta_1 = (2n + 1)\frac{\pi}{2}$, where n is any integer.

Now $$p = \lim_{\theta\to\theta_1} -\frac{d\theta}{du} = \lim_{\theta\to\theta_1} \frac{(a + b\sin\theta)^2}{a\sin\theta + b}$$

$$= \frac{\left[a + b\sin(2n+1)\frac{\pi}{2}\right]^2}{a\sin(2n+1)\frac{\pi}{2} + b} = \frac{\left[a + b\sin\left(n\pi + \frac{\pi}{2}\right)\right]^2}{a\sin\left(n\pi + \frac{\pi}{2}\right) + b}$$

$$= \frac{\left[a + b(-1)^n \sin\frac{\pi}{2}\right]^2}{a(-1)^n \sin\frac{\pi}{2} + b} = \frac{\left[a + b(-1)^n\right]^2}{a(-1)^n + b}$$

The equation of asymptotes is

$$p = r\sin(\theta_1 - \theta)$$

or $$\frac{\left[a + b(-1)^n\right]^2}{a(-1)^n + b} = r\sin\left[(2n+1)\frac{\pi}{2} - \theta\right]$$

$$= r\sin\left(n\pi + \frac{\pi}{2} - \theta\right) = (-1)^n r\sin\left(\frac{\pi}{2} - \theta\right)$$

$$= (-1)^n r\cos\theta.$$

Example 2 : Find the asymptotes of $r^n f_n(\theta) + r^{n-1} f_{n-1}(\theta) + \ldots + f_0(\theta) = 0$.

Solution : Changing r into $\frac{1}{u}$, the given equation becomes

$$f_n(\theta) + uf_{n-1}(\theta) + u^2 f_{n-2}(\theta) + \ldots + u^n f_0(\theta) = 0 \qquad (1)$$

Remarks

Now $u \to 0$ gives $f_n(\theta) \to 0$

Let θ_1 be any root of $f_n(\theta) = 0$

Differentiating (1) w.r.t. u, we get

$$f_n'(\theta) + f_{n-1}(\theta)\frac{du}{d\theta} + uf_{n-1}'(\theta) + (\text{terms containing power of } u) = 0$$

Taking u = 0, we get

$$f_n'(\theta) + f_{n-1}(\theta)\frac{du}{d\theta} = 0$$

$$\frac{du}{d\theta} = -\frac{f_n'(\theta)}{f_{n-1}(\theta)}$$

$\therefore$ $$p = \lim_{\theta\to\theta_1} -\frac{d\theta}{du} = \frac{f_{n-1}(\theta_1)}{f_n'(\theta_1)}$$

Thus the equation of asymptote is $p = r\sin(\theta_1 - \theta)$

$$\frac{f_{n-1}(\theta_1)}{f_n'(\theta_1)} = r\sin[\theta_1 - \theta]$$

where θ_1 is any root of equation $f_n(\theta) = 0$.

Remark : The reault for asymptotes of polar curves can also be expressed as :

If α is a root of the equation $f(\theta) = 0$, then $r\sin(\theta - \alpha) = \dfrac{1}{f'(\alpha)}$ is an asymptote of the curve $f(\theta)$.

Example 3 : Find the asymptotes of the curve $r = \dfrac{2a}{1 + 2\cos\theta}$.

Solution : Putting, $u = \dfrac{1}{r}$, the equation of the curve becomes

$$2au = 1 + 2\cos\theta \qquad (1)$$

When $u \to 0$, then $$\cos\theta_1 = -\frac{1}{2} = \cos\frac{2\pi}{3}$$

$\therefore$ $$\theta_1 = 2n\pi \pm \frac{2\pi}{3};\ n = 0, \pm 1, \pm 2, \pm 3 \ldots.$$

Differentiating (1) w.r.t θ, we have

$$2a\frac{du}{d\theta} = -2\sin\theta$$

$$\frac{d\theta}{du} = -a\,\mathrm{cosec}\,\theta$$

$\therefore$ $$p = \lim_{\theta\to\theta_1} -\frac{d\theta}{du} = \lim_{\theta\to\theta_1} a\,\mathrm{cosec}\,\theta$$

Remarks

$$= a\,\mathrm{cosec}\left(2n\pi \pm \frac{2\pi}{3}\right) = \pm a\,\mathrm{cosec}\frac{2\pi}{3} = \pm\frac{2a}{\sqrt{3}}$$

The equation of asymptotes is $p = r\sin(\theta_1 - \theta)$

$$\pm\frac{2a}{\sqrt{3}} = r\sin\left[2n\pi \pm \frac{2\pi}{3} - \theta\right] = r\sin\left(\pm\frac{2\pi}{3} - \theta\right)$$

$\therefore$
$$\frac{2a}{\sqrt{3}} = r\sin\left(\frac{2\pi}{3} \mp \theta\right)$$

Hence, the asymptotes are $\dfrac{2a}{\sqrt{3}} = r\sin\left(\dfrac{2\pi}{3} \mp \theta\right)$

Example 4 : Find the asymptotes of the curve $r\cos 2\theta = a\sin 3\theta$.

Solution : The given equation of curve is $r = \dfrac{a\sin 3\theta}{\cos 2\theta}$

Putting $r = \dfrac{1}{u}$, we get $u = \dfrac{\cos 2\theta}{a\sin 3\theta}$

(1)

Now $u \to 0$ gives $\cos 2\theta \to 0$

$\therefore$
$$2\theta_1 \to (2n+1)\frac{\pi}{2}$$

$$\theta_1 \to (2n+1)\frac{\pi}{4}$$ where n is any integer.

Differentiating (1) w.r.t θ, we get

$$\frac{du}{d\theta} = \frac{-2\sin 2\theta\sin 3\theta - 3\cos 3\theta\cos 2\theta}{a\sin^2 3\theta}$$

$\therefore$
$$p = \lim_{\theta\to\theta_1} -\frac{d\theta}{du} = \frac{a\sin^2 3\theta_1}{2\sin 2\theta_1\sin 3\theta_1 + 3\cos 3\theta_1\cos 2\theta_1}$$

(2)

Now
$$\sin 2\theta_1 = \sin 2(2n+1)\frac{\pi}{4} = \sin\left(n\pi + \frac{\pi}{2}\right) = (-1)^n\sin\frac{\pi}{2} = (-1)^n$$

$$\cos 2\theta_1 = \cos 2(2n+1)\frac{\pi}{4} = \cos\left(n\pi + \frac{\pi}{2}\right) = (-1)^n\cos\frac{\pi}{2} = 0$$

$$\sin 3\theta_1 = \sin 3(2n+1)\frac{\pi}{4} = \sin\left(3\frac{n\pi}{2} + \frac{3\pi}{4}\right)$$

$$= \sin\left(n\pi + \frac{n\pi}{2} + \frac{3\pi}{4}\right) = (-1)^n\sin\left(\frac{n\pi}{2} + \frac{3\pi}{4}\right)$$

Remarks

From(2), $$p = \frac{a\sin^2 3\theta_1}{2\sin 2\theta_1 \sin 3\theta_1}$$

$$= \frac{a\sin 3\theta_1}{2\sin 2\theta_1} = \frac{a(-1)^n \sin\left[\frac{n\pi}{2} + \frac{3\pi}{4}\right]}{2(-1)^n}$$

$\therefore$ Equation of asymptote is $p = r\sin(\theta_1 - \theta)$

i.e., $$\frac{a}{2}\sin\left(\frac{n\pi}{2} + \frac{3\pi}{4}\right) = r\sin\left[(2n+1)\frac{\pi}{4} - \theta\right]$$

Exercise 2.7

Find the asymptotes of the following curves :

1. (i) $r\cos\theta = a\cos 2\theta$ (ii) $r\sin\theta = 2\cos 2\theta$ (iii) $r\cos\theta = a\sin^2\theta$
2. (i) $r(1 - e^\theta) = a$ (ii) $r(\pi + \theta) = ae^\theta$
3. (i) $r = \dfrac{3a\sin\theta\cos\theta}{\sin^3\theta + \cos^3\theta}$ (ii) $r^n \sin n\theta = a^n$
4. (i) $r = a\operatorname{cosec}\theta + b\cot\theta$ (ii) $r = a + b\cot n\theta$ (iii) $r = a\operatorname{cosec}\theta + b$
5. (i) $r\cos\theta = a\sin\theta$ or $r = a\tan\theta$ (ii) $r\theta\cos\theta = a\cos 2\theta$
   (iii) $r\sin n\theta = a$ (iv) $r\sin 2\theta = a$

Answers

1. (i) $r\cos\theta + a = 0$ (ii) $r\sin\theta = 2$
   (iii) $a = r\cos\theta$
2. (i) $r\sin\theta + a = 0$ (ii) $r\sin\theta . e^\pi + a = 0$
3. (i) $\sqrt{2}r\sin\left(\frac{\pi}{4} + \theta\right) + a = 0$ (ii) $\theta = \frac{p\pi}{n}, p \in I$
4. (i) $a(-1)^n + b = (-1)^n r\sin\theta$ (ii) $r\sin\left(\theta - \frac{m\pi}{n}\right) = \frac{b}{n}, m \in I$ (iii) $r\sin\theta = a$
5. (i) $r\cos\theta = (-1)^n a$ (ii) $r\sin\theta = a$, $(2n+1)\pi r\cos\theta + 2a = 0$
   (iii) $r\sin\left(\theta - \frac{m\pi}{n}\right) = a\frac{(-1)^m}{n}, m \in I$ (iv) $r\sin\theta = \pm\frac{a}{2}, r\cos\theta = \pm\frac{a}{2}$

Keywords : Taylor's theorem, Taylor's theorem with Cauchy form of remainder after n terms, asymptote || x-axis, asymptote || y-axis, oblique asymptote, asymptote of polar curves.

Remarks

Summary

A series is a set of terms which are arranged according to some fixed definite law connected by +ve or –ve signs. The series may be of two types viz. finite series and infinite series according as the number of terms it contains are finite or infinite. There are 3 types of asymptotes.

1. Asymptote || x-axis.
2. Asymptote || y-axis.
3. Oblique asymptote.

Remarks

CHAPTER – III
CURVATURE

3.0 STRUCTURE

3.1 INTRODUCTION

We study the rate of change of the angle ψ w.r.t. the rate of change of arc length. The ratio $\frac{\delta\psi}{\delta S}$ represents the average rate of change in the angle ψ per unit of arc length along the curve and is called the average curvature of arc PQ.

$$\text{Curvature} = \lim_{\delta S\to 0}\frac{\delta\psi}{\delta S} = \frac{d\psi}{dS} = \frac{1}{\delta} \qquad \text{where } \delta = \text{radius of curvature.}$$

3.2 OBJECTIVE

After reading this lesson, you should be able to understand

- Curvature
- Radius of curvature
- Centre of curvature & circle of curvature
- Evolute

3.3 DEFINITION :

The curvature of a curve C at a point (x, y) on C is denoted by the Greek letter κ (kappa) and is given by the equation

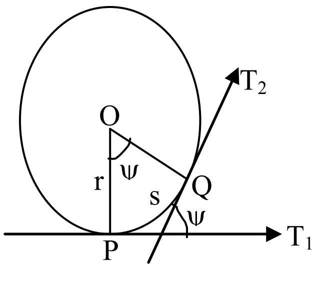
$$\kappa = \left|\frac{d\psi}{ds}\right|$$

where ψ is the angle which the tangent line to C at (x, y) makes with the positive x-axis and s is the arc length as measured along the curve.

Remark : A straight line does not bend at all (as ψ is constant so $\frac{d\psi}{ds}$ is zero). Hence the curvature of a straight line is zero.

3.4 CURVATURE OF CIRCLE

To prove that the curvature of a circle at any point on it is constant and is equal to the reciprocal of the radius of the circle.



Let there be a circle of radius r with centre at O. Let P and Q be two points on the circle, so that arc PQ = s. Let the angle between the tangents at P and Q be ψ. Then $\angle POQ = \psi$. Using the formula

$l = r\theta$, we have

Remarks

$$s = r\psi$$

Differentiating w.r.t. ψ, we get

$$\frac{ds}{d\psi} = r$$

$\therefore$ Curvature $= \dfrac{d\psi}{ds}$

$$= \frac{1}{r} \text{ (constant)}$$

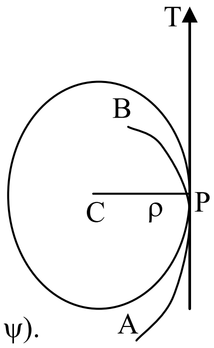
(1)

3.5 RADIUS OF CURVATURE

The radius of curvature of a given curve, at a given point on it, is the radius of the circle, whose curvature is equal to the curvature of the curve at that point.

Let P be any point on the curve AB.

Let the circle with centre C and radius ρhas the same curvature as that of the curve at point P. Then the curvature of the curve at P = the curvature of the circle



i.e., $\dfrac{d\psi}{ds} = \dfrac{1}{\rho}$

or $\rho = \dfrac{ds}{d\psi}$

Example 1 : Find the radius of curvature for s = log(sec ψ + tan ψ).

Solution : $s = \log(\sec\psi + \tan\psi)$

$$\Rightarrow \frac{ds}{d\psi} = \frac{1}{\sec\psi + \tan\psi}(\sec\psi\tan\psi + \sec^2\psi)$$

or $\rho = \sec\psi$.

Formulae to find radius of curvature

1. Cartesian Equation $y = f(x)$

$$\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}, \qquad y_1 = \frac{dy}{dx}, \; y_2 = \frac{d^2y}{dx^2}$$

2. Parametric Equation $x = f(t)$,
$y = g(t)$

$$\rho = \frac{\left(x'^2 + y'^2\right)^{\frac{3}{2}}}{x'y'' - x''y'} \qquad \text{where}$$

$$x' = \frac{dx}{dt}, y' = \frac{dy}{dt}$$

$$x'' = \frac{d^2x}{dt^2}, y'' = \frac{d^2y}{dt^2}$$

3. Polar Equation $r = f(\theta)$

$$\rho = \frac{\left(r^2 + r_1^2\right)^{\frac{3}{2}}}{r^2 + 2r_1^2 - rr_2}$$

Remarks

where $$r_1 = \frac{dr}{d\theta},\ r_2 = \frac{d^2r}{d\theta^2}$$

4. Pedal Equation $P = f(r)$

$$\rho = r\frac{dr}{dP}$$

Example 1 : Show that in the parabola $y^2 = 4ax$, ρ the radius of curvature at any point P is twice the part of the normal intercepted between the curve and the directrix. Also prove that ρ^2 varies as $(SP)^3$ where S is the focus.

Solution : The curve is $y^2 = 4ax \Rightarrow y = \sqrt{a}\sqrt{x}$

$\therefore$ $$y_1 = 2\sqrt{a}.\frac{1}{2\sqrt{x}} = \frac{\sqrt{a}}{\sqrt{x}}$$

and $$y_2 = -\frac{\sqrt{a}}{2x^{3/2}}$$

$\therefore$ $$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{\left(1+\frac{a}{x}\right)^{3/2}}{-\frac{\sqrt{a}}{2x^{3/2}}}$$

$$= \frac{2(x+a)^{3/2}}{\sqrt{a}} \text{ (in magnitude)}$$

(2)

Slope of the normal $= -\frac{1}{y_1} = -\frac{\sqrt{x}}{\sqrt{a}}$

and the equations of the normal at P is $Y - y = -\frac{\sqrt{x}}{\sqrt{a}}(X - x)$

It meets the directrix $X = -a$ at $Y = y - \frac{\sqrt{x}}{\sqrt{a}}(-a - x)$

or

$$Y = \frac{\sqrt{x}}{\sqrt{a}}(x + 3a) \qquad \left[\because y = 2\sqrt{a}\sqrt{x}\right]$$

$\therefore$ The point where the normal at P meets the directrix $x = -a$ is $\left[-a, \frac{\sqrt{x}}{\sqrt{a}}(x+3a)\right]$ and the distance between this point and P is

$$\left[(x+a)^2 + \left\{\frac{\sqrt{x}}{\sqrt{a}}(x+3a) - 2\sqrt{a}\sqrt{x}\right\}^2\right]^{\frac{1}{2}} = \frac{(x+a)^{3/2}}{\sqrt{a}} \text{ (on simplification)} \qquad (3)$$

From (2) and (3), we have ρ = twice this distance.
Hence the result.
Again S being the focus, its coordinates are (a, 0)

Remarks

$\therefore \qquad SP = \sqrt{(x-a)^2 + y^2} = \sqrt{(x-a)^2 + 4ax} = x + a$ (4)

$\therefore \qquad \rho^2 \propto (SP)^3$ [From (2)]

Example 2 : If ρ_1 and ρ_2 are the radii of curvature at the extremities of a focal chord of a parabola whose semi latus-rectum is l, prove that $(\rho_1)^{-2/3} + (\rho_2)^{-2/3} = (l)^{-2/3}$.

Solution : Let the equation of parabola be

$$y^2 = 4ax$$

Its parametric equations are

$$x = at^2,\ y = 2at \qquad (1)$$

Also, semi latus-rectum = $2a = l$ (given)

From (1), $\quad x' = 2at, \quad y' = 2a,$

$\quad x'' = 2a, \quad y'' = 0,$

where dashes denote differentiation w.r.t. parameter t.

$$\therefore \qquad \rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - x''y'}$$

$$= \frac{(4a^2t^2 + 4a^2)^{3/2}}{0 - 4a^2} = \frac{-8a^3(1+t^2)^{3/2}}{4a^2}$$

$= 2a(1 + t^2)^{3/2}$ (numerically)

$$= l(1 + t^2)^{3/2} \qquad (2)$$

Let $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ be the extremities of focal chord of the parabola, then $t_1t_2 = -1$.

Also, ρ_1(radius of curvature at P) $= l(1 + t_1^2)^{3/2}$ [From (2)]

ρ_2 (radius of curvature at Q) $= l(1 + t_2^2)^{3/2}$

$$\therefore \qquad (\rho_1)^{-2/3} + (\rho_2)^{-2/3} = l^{-2/3}\left[(1+t_1^2)^{-1} + (1+t_2^2)^{-1}\right]$$

$$= l^{-2/3}\left[\frac{1}{1+t_1^2} + \frac{1}{1+t_2^2}\right]$$

$$= l^{-2/3}\left[\frac{1}{1+t_1^2} + \frac{1}{1+\frac{1}{t_1^2}}\right] \qquad [\because t_1t_2 = -1]$$

$$= l^{-2/3}.\left(\frac{1+t_1^2}{1+t_1^2}\right) = l^{-2/3}$$

Example 3 : If ρ_1, ρ_2 be radii of curvature at the extremities of a pair of semi-conjugate diameters of an ellipse, prove that $\left[(\rho_1)^{2/3} + (\rho_2)^{2/3}\right](ab)^{2/3} = a^2 + b^2$.

Solution : Let the ellipse be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Its parametric equations are $\quad x = a\cos\theta, \quad y = b\sin\theta$

$\therefore \qquad x' = -a\sin\theta, \quad y' = b\cos\theta$

$\qquad x'' = -a\cos\theta, \quad y'' = -b\sin\theta$

Remarks

$$\therefore \qquad \rho(a\cos^3\theta, b\sin^3\theta) = \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{ab\sin^2\theta + ab\cos^2\theta}$$

$$= \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{ab}$$

(1)

Let CP and CD be two semi-conjugate* diameters. If the co-ordinates of P are (a cos θ, b sin θ), then co-ordinates of D are

$$\left[a\cos\left(\frac{\pi}{2}+\theta\right), b\sin\left(\frac{\pi}{2}+\theta\right)\right]$$

If ρ_1 is the radius of curvature at P, then

$$\rho_1 = \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{ab}$$

Changing $\theta \to \left(\frac{\pi}{2}+\theta\right)$, we get

ρ_2, the radius of curvature at $D = \dfrac{(a^2\cos^2\theta + b^2\sin^2\theta)^{3/2}}{ab}$

$$\therefore \qquad (\rho_1)^{2/3} + (\rho_2)^{2/3} = \frac{a^2\sin^2\theta + b^2\cos^2\theta}{(ab)^{2/3}} + \frac{a^2\cos^2\theta + b^2\sin^2\theta}{(ab)^{2/3}}$$

or $$[(\rho_1)^{2/3} + (\rho_2)^{2/3}](ab)^{2/3} = a^2(\sin^2\theta + \cos^2\theta) + b^2(\cos^2\theta + \sin^2\theta)$$

$$= a^2 + b^2.$$

Example 4 : Prove that for the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$ or the curve

$x^{2/3} + y^{2/3} = a^{2/3}at(a\cos^3\theta,\ a\sin^3\theta)$; $\qquad \rho = 3a\sin\theta\cos\theta$.

Solution : The equation of the curve is

$$x = a\cos^3\theta,\ y = a\sin^3\theta$$

(1)

$$\therefore \qquad x' = \frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta,$$

$$y' = \frac{dy}{d\theta} = 3a\sin^2\theta.\cos\theta$$

Also, $\qquad x'' = 3a\cos\theta(2\sin^2\theta - \cos^2\theta),$

$$y'' = 3a\sin\theta(2\cos^2\theta - \sin^2\theta),$$

where dashes denote differentiation w.r.t. parameter 't'.

$$\therefore \qquad \rho = \frac{(x'^2 + y'^2)^{3/2}}{[x'y'' - x''y']}$$

$$= \frac{(9a^2\sin^2\theta\cos^2\theta)^{3/2}(\cos^2\theta + \sin^2\theta)^{3/2}}{9a^2\sin^2\theta\cos^2\theta[-2\cos^2\theta + \sin^2\theta - 2\sin^2\theta + \cos^2\theta}$$

$$= -(9a^2\sin^2\theta\cos^2\theta)^{1/2}(\cos^2\theta + \sin^2\theta)^{1/2}$$

$= 3a\sin\theta\cos\theta$ (numberically).

Exercise 3.1

1. Find the radius of curvature at (s, ψ) for the curve $s = a\log\tan\left(\frac{\pi}{4}+\frac{\psi}{2}\right)$.

Remarks

2. Find the radius of curvature at the given point $\left(\frac{1}{4},\frac{1}{4}\right)$ for the curve $\sqrt{x}+\sqrt{y}=1$.
3. Prove that for the curve $x^3 + y^3 - 3axy = 0$, the radius of curvature at the point $\left(\frac{3a}{2},\frac{3a}{2}\right)$ is $\frac{3a}{8\sqrt{2}}$ numerically.
4. Find the radius of curvature at any point for the curves $x = a(\cos t + t \sin t)$, $y = a(\sin t - t \cos t)$.
5. Show that the radius of curvature at a point P of the curve $x = a\left(\cos t + \log \tan \frac{t}{2}\right)$, $y = a \sin t$ is inversely proportional to the length of the normal intercepted between the point P and the x-axis.
6. The tangents at two points P, Q on the cycloid $x = a(\theta + \sin\theta)$; $y = a(1 - \cos\theta)$ are at right angles. Show that if ρ_1, ρ_2 are the radii of curvature at these points then $\rho_1^2 + \rho_2^2 = 16a^2$.
7. Find a point on the parabola $y^2 = 8x$ at which the radius of curvature is $7\frac{13}{16}$.
8. Prove that for the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

(i) $\rho = \frac{a^2b^2}{p^3}$, where p is the perpendicular from the centre upon the tangent at (x,y).

(ii) If CP, CD be a pair of semi-conjugate diameters of an ellipse of semi-axes of length a and b, prove that the radius of curvature at $P = \frac{CD^3}{ab}$, where C is the centre of the ellipse.

(iii) Show that the radius of curvature at any point P(x, y) of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ is given by $\rho^3 = 27\,axy$.

9. Show that in an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the radius of curvature at the end of the major axis is equal to the semi latus-rectum of the ellipse.
10. Show that for the curve $y = \log x$, the least value of $|\rho|$ is $\frac{3\sqrt{3}}{2}$.

Answers

1. $a \sec\psi$ 2. $\frac{1}{\sqrt{2}}$ 4. at 7. $\left(\frac{9}{8},\pm 3\right)$.

3.6 TO FIND ρ IF THE EQUATION OF THE CURVE IS IN THE FORM U = F(θ) WHERE $\mathbf{u} = \frac{1}{\mathbf{r}}$.

Since $$r = \frac{1}{u}$$

$\therefore$ $$r_1 = -\frac{1}{u^2}.u_1, \quad \text{where} \quad u_1 = \frac{du}{d\theta}$$

and $$r_2 = -\frac{u^2u_2 - u_1.2uu_1}{u^4} = \frac{-2u_1^2 - uu_2}{u^3}$$

Remarks

$$\therefore \qquad \rho = \frac{\left(\frac{1}{u^2} + \frac{u_1^2}{u^4}\right)^{\frac{3}{2}}}{\frac{2u_1^2}{u^4} + \frac{1}{u^2} - \frac{2u_1^2 - uu_2}{u^4}}$$

$$\rho = \frac{(u^2 + u_1^2)^{\frac{3}{2}}}{u^3(u + u_2)}.$$

Example 1 : Find the radius of curvature for the parabola $\frac{2a}{r} = 1 + \cos\theta$. Hence show that the square of the radius of curvature varies as the cube of the focal distance.

Solution : The given equation of parabola is

$$\frac{2a}{r} = 1 + \cos\theta \qquad (1)$$

Putting $u = \frac{1}{r}$ in (1), we have

$$2au = 1 + \cos\theta \qquad (2)$$

Differentiating (2) w.r.t. θ, we have

$$2au_1 = -\sin\theta \qquad (3)$$

Differentiating (3) w.r.t. θ, we have

$$2au_2 = -\cos\theta$$

$$\therefore \qquad \rho = \frac{(u^2 + u_1^2)^{3/2}}{u^3(u + u_2)}$$

$$= \frac{\left[\left(\frac{1+\cos\theta}{2a}\right)^2 + \frac{\sin^2\theta}{4a^2}\right]}{\left(\frac{1+\cos\theta}{2a}\right)^3\left[\frac{1+\cos\theta}{2} - \frac{\cos\theta}{2a}\right]}$$

$$= \frac{\frac{1}{8a^3}[2(1+\cos\theta)]^{3/2}}{\frac{1}{16a^4}(1+\cos\theta)^3} = \frac{4\sqrt{2}a}{(1+\cos\theta)^{3/2}} = \frac{2}{\sqrt{a}}r^{3/2}$$

$$\therefore \qquad \rho^2 = \frac{4}{a}r^3$$

In the given equation of the parabola i.e., $\frac{2a}{r} = 1 + \cos\theta$, the pole is at the focus. Therefore the focal distance of the point on the parabola is r. Hence the square of the radius of curvature varies as the cube of the focal distance.

Remarks

Remark : The radius of curvature for polar curves can also be found by first changing the polar equation of the curve to pedal form.

3.7 RULE TO TRANSFORM POLAR EQUATION TO PEDAL EQUATION

1. Use the relation $\tan\phi = r\dfrac{d\theta}{dr}$ and find ϕ.
2. Put this value of ϕ in $p = r \sin \phi$.
3. Eliminate θ to get the equation in Pedal form.

The following example will illustrate the above method :

Example 2 : Find the radius of curvature for the curve $r^n = a^n \cos n\theta$.

Solution : The equation of the curve is

$$r^n = a^n \cos n\theta \qquad (1)$$

[Polar form]

$$\therefore \quad n \log r = n \log a + \log \cos n\theta$$

Differentiating, $\dfrac{n}{r}.\dfrac{dr}{d\theta} = \dfrac{1}{\cos n\theta}(-n\sin n\theta)$

or
$$r\frac{d\theta}{dr} = -\cot n\theta$$

Also
$$\tan\phi = r\frac{d\theta}{dr} = -\cot n\theta$$

or
$$\tan\phi = \tan\left(\frac{\pi}{2} + n\theta\right)$$

i.e.,
$$\phi = \frac{\pi}{2} + n\theta$$

(2)

Now
$$p = r\sin\phi = r\sin\left(\frac{\pi}{2} + n\theta\right)$$
$$= r\cos n\theta = r.\frac{r^n}{a^n}$$

$\therefore$ Pedal equation of the given curve is

$$p = \frac{r^{n+1}}{a^n}$$

Differentiating, $\dfrac{dp}{dr} = \dfrac{(n+1)}{a^n}r^n$

$$\therefore \quad \rho = r\frac{dr}{dp} = r.\frac{a^n}{(n+1)r^n} = \frac{a^n}{(n+1)r^{n-1}}.$$

Example 3 : Find the radius of curvature for the hyperbola $r^2 = a^2 - b^2 + \dfrac{a^2b^2}{p^2}$.

Solution : The given equation is

$$r^2 = a^2 - b^2 + \frac{a^2b^2}{p^2} \qquad (1)$$

Differentiating (1) w.r.t. p, we get

Remarks

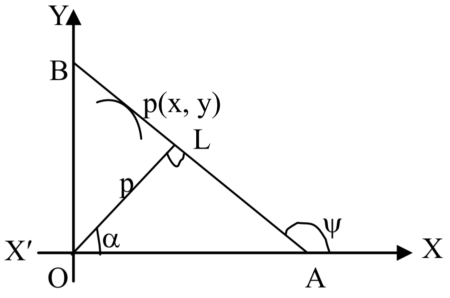
$$2r\frac{dr}{dp} = -\frac{2a^2b^2}{p^3}$$

$\therefore$ $$\rho = r\frac{dr}{dp} = \frac{a^2b^2}{p^3} \text{ (numerically).}$$

3.8 RADIUS OF CURVATURE OF THE POLAR TANGENTIAL EQUATIONS P = F(ψ).

A relation between p and ψ with usual notations is called the polar tangential or tangential polar equation.

From figure, we have



$$\psi = \alpha + \frac{\pi}{2} \tag{1}$$

Also the equation of the line AB in normal form is

$$X \cos \alpha + Y \sin \alpha = p \tag{2}$$

$\therefore$ From (1) and (2),

$$p = X\cos\left(\psi - \frac{\pi}{2}\right) + Y\sin\left(\psi - \frac{\pi}{2}\right)$$

$$= X \sin \psi - Y \cos \psi$$

Since P(x, y) lies on this line

$\therefore$ $$p = x \sin \psi - y \cos \psi \tag{3}$$

Differentiating both sides w.r.t. ψ, we have

$$\frac{dp}{d\psi} = x\cos\psi + \sin\psi.\frac{dx}{d\psi} + y\sin\psi - \cos\psi.\frac{dy}{d\psi}$$

$$= x\cos\psi + y\sin\psi + \sin\psi\frac{dx}{ds}.\frac{ds}{d\psi} - \cos\psi.\frac{dy}{ds}.\frac{ds}{d\psi}$$

$$= x \cos \psi + y \sin \psi + \sin\psi.\cos\psi.\rho - \cos\psi \sin\psi.\rho$$

$$\left[\because \frac{dx}{ds} = \cos\psi \text{ and } \frac{dy}{ds} = \sin\psi\right]$$

$$= x \cos \psi + y \sin \psi$$

Again differentiating w.r.t. ψ, we have

$$\frac{d^2p}{d\psi^2} = -x\sin\psi + \cos\psi.\frac{dx}{d\psi} + y\cos\psi + \sin\psi.\frac{dy}{d\psi}$$

$$= -(x\sin\psi - y\cos\psi) + \cos\psi\frac{dx}{ds}.\frac{ds}{d\psi} + \sin\psi.\frac{dy}{ds}.\frac{ds}{d\psi}$$

$\therefore$ $$\rho = p + \frac{d^2p}{d\psi^2}$$

Example 4 : Find the radius of curvature for the ellipse $p^2 = a^2 \cos^2\psi + b^2 \sin^2\psi$.

Solution : The given equation is $p^2 = a^2 \cos^2\psi + b^2 \sin^2\psi$ (1)

Remarks

Differentiating both sides of (1) w.r.t ψ

$$2p\frac{dp}{d\psi} = -2a^2 \cos\psi \sin\psi + 2b^2 \sin\psi \cos\psi$$

$$= 2(b^2 - a^2)\cos\psi \sin\psi$$

Cancelling the common factor 2 and again differentiating with respect to ϕ, we have

$$p.\frac{d^2p}{d\psi^2} + \left(\frac{dp}{d\psi}\right)^2 = (b^2 - a^2)(\cos^2\psi - \sin^2\psi)$$

or

$$p\frac{d^2p}{d\psi^2} = (b^2 - a^2)(\cos^2\psi - \sin^2\psi) - \frac{(b^2 - a^2)\cos^2\psi\sin^2\psi}{p^2}$$

$$= \frac{(b^2 - a^2)[(a^2\cos^2\psi + b^2\sin^2\psi)(\cos^2\psi - \sin^2\psi) - (b^2 - a^2)\cos^2\psi\sin^2\psi]}{p^2} \text{ or}$$

$$\frac{d^2p}{d\psi^2} = \frac{(b^2 - a^2)[a^2\cos^4\psi - b^2\sin^4\psi]}{p^3}$$

$\therefore$

$$\rho = p + \frac{d^2p}{d\psi^2} = \frac{p^4 + (b^2 - a^2)(a^2\cos^4\psi - b^2\sin^4\psi)}{p^3}$$

$$= \frac{(a^2\cos^2\psi + b^2\sin^2\psi)^2 + (b^2 - a^2)(a^2\cos^4\psi - b^2\sin^4\psi)}{p^3}$$

$$= \frac{a^2b^2}{p^3}$$

Exercise 3.2

1. If ρ_1 and ρ_2 are the radii of curvature at the extremities of any chord through the pole of the cardioide r = a(1 – cosθ), show that $\rho_1^2 + \rho_2^2 = \frac{16a^2}{9}$.
2. Show that for the cardioid r = a(1 + cosθ), the radius of curvature
$$\rho = \frac{4a}{3}\cos\frac{\theta}{2} \text{ and } \frac{\rho^2}{r} = \frac{8a}{9}.$$
3. Show that the radius of curvature to r = a sin nθ where r = a is $\frac{a}{1+n^2}$.
4. Find the radius of curvature for the curve $r^2 \cos 2\theta = a^2$.
5. Find the radius of curvature at (r, θ) of the conic $\frac{1}{r} = 1 + e\cos\theta$.

Answers

4. $\frac{r^3}{a^2}$ 5. $\frac{l(1 + 2e\cos\theta + e^2)^{3/2}}{(1 + e\cos\theta)^3}$.

3.9 CENTRE OF CURVATURE, CIRCLE OF CURVATURE AND EVOLUTE

We recall the definition of article in which the point C is the centre of curvature and the circle with centre C and radius ρ is called the circle of curvature corresponding to point P of the curve. Again for different positions of P along the curve, C also takes up different positions. The locus of C is thus called the evolute of the curve. The curve itself is called the involute of its evolute. We now proceed to find these for different curves.

Remarks

1. Centre of curvature (Cartesian form)

If (X, Y) = centre of curvature

Then

$$X = x - \frac{y_1(1+y_1^2)}{y_2}, \qquad y_1 = \frac{dy}{dx}, y_2 = \frac{d^2y}{dx^2}$$

$$Y = y + \frac{1+y_1^2}{y_2}$$

2. Circle of curvature

If (X, Y) = centre of curvature &ρ = radius of curvature

Then equation of circle of curvature is

$$(x - X)^2 + (y - Y)^2 = \rho^2$$

Chord of Curvature

1. Cartesian equations

(a) parallel to x-axis

$$C_x = \frac{2y_1(1+y_1^2)}{y_2}$$

(b) Parallel to y-axis

$$C_y = \frac{2(1+y_1^2)}{y_2}$$

2. Polar equations

(a) Through Pole

$$C_O = \frac{2r(r^2+r_1^2)}{r^2+2r_1^2-rr_2}$$

(b) Perpendicular to radius vector

$$C_p = \frac{2r_1(r^2+r_1^2)}{r^2+2r_1^2-rr_2}$$

3. Pedal equations

(a) $$C_O = 2p\frac{dr}{dp}$$

(b) $$C_P = 2\sqrt{r^2-p^2}\,\frac{dr}{dP}$$

Example 1 : If C_x, C_y be the chords of curvature parallel to co-ordinates axes at any point of the curve $y = c\cosh\frac{x}{c}$; prove that $4c^2(C_x^2 + C_y^2) = C_y^4$.

Solution : The given curve is $y = c\ \cosh\frac{x}{c}$ (1)

Differentiating (1) w.r.t. x, we have

$$y_1 = c\sinh\frac{x}{c}.\frac{1}{c} = \sinh\frac{x}{c} \text{ and } y_2 = \frac{1}{c}\cosh\frac{x}{c}$$

Remarks

$$\therefore \quad C_x = \frac{2y_1(1+y_1^2)}{y_2} = \frac{2\sinh\frac{x}{c}\left(1+\sinh^2\frac{x}{c}\right)}{\frac{1}{c}\cosh\frac{x}{c}}$$

$$= 2c\sinh\frac{x}{c}\cosh\frac{x}{c} \qquad \left[\because 1+\sinh^2\frac{x}{c} = \cosh^2\frac{x}{c}\right]$$

and $$C_y = \frac{2(1+y_1^2)}{y_2} = \frac{2\left(1+\sinh^2\frac{x}{c}\right)}{\frac{1}{c}\cosh\frac{x}{c}} = 2c\cosh\frac{x}{c}$$

(2)

$$\therefore \quad \text{L.H.S.} = 4c^2(C_x^2 + C_y^2)$$

$$= 4c^2\left[4c^2\sinh^2\frac{x}{c}\cosh^2\frac{x}{c} + 4c^2\cosh^2\frac{x}{c}\right]$$

$$= 16c^4\cosh^2\frac{x}{c}\left(\sinh^2\frac{x}{c}+1\right) = 16c^4\cosh^4\frac{x}{c}$$

$$= \left(2c\cosh\frac{x}{c}\right)^4 = C_y^4 = \text{R.H.S.}$$

Hence the result.

Example 2 : Find the coordinates of the centre of curvature at any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ or $x = a\cos\theta$, $y = b\sin\theta$. Hence find the equation of its evolute.

Solution : It would be better to take the parametric equations of the ellipse i.e. $x = a\cos\theta$, $y = b\sin\theta$.

$$\therefore \quad y_1 = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cot\theta$$

and $$y_2 = \frac{b}{a}\operatorname{cosec}^2\theta.\frac{d\theta}{dx} = \frac{b}{a}\operatorname{cosec}^2\theta.\frac{1}{-a\sin\theta} = -\frac{b}{a^2}\operatorname{cosec}^3\theta$$

$$\therefore \quad X = X - \frac{y_1(1+y_1^2)}{y_2}$$

$$= a\cos\theta - \frac{-\frac{b}{a}\cot\theta\left(1+\frac{b^2}{a^2}\cot^2\theta\right)}{-\frac{b}{a^2}\operatorname{cosec}^3\theta}$$

Remarks

$$= a\cos\theta - a\cos\theta\sin^2\theta - \frac{b^2}{a}\cos^3\theta$$

$$= a\cos\theta(1-\sin^2\theta) - \frac{b^2}{a}\cos^3\theta$$

$$= \frac{a^2-b^2}{a}\cos^3\theta \qquad (1)$$

$$Y = y + \frac{1+y_1^2}{y_2} = b\sin\theta + \frac{1+\frac{b^2}{a^2}\cot^2\theta}{-\frac{b}{a^2}\operatorname{cosec}^3\theta}$$

$$= b\sin\theta - \frac{a^2}{b}\sin^3\theta - b\sin\theta\cos^2\theta$$

$$= b\sin\theta(1-\cos^2\theta) - \frac{a^2}{b}\sin^3\theta = -\frac{a^2-b^2}{b}\sin^3\theta \qquad (2)$$

Now (1) and (2) give co-ordinates of the centre of curvature in terms of θ which are the parametric equations of the evolute.

Eliminating θ from (1) and (2) we get the Cartesian equation of the evolute of the ellipse.

From (1), $aX = (a^2 - b^2)\cos^3\theta$

or $(aX)^{2/3} = (a^2 - b^2)^{2/3}\cos^2\theta$ (3)

Similarly from (2) $(bY)^{2/3} = (a^2 - b^2)^{2/3}\sin^2\theta$ (4)

Adding (3) and (4), we get $(aX)^{2/3} + (bY)^{2/3} = (a^2 - b^2)^{2/3}$, which is the equation of the evolute.

Exercise 3.3

1. Find the centre of curvature for the curve $x^3 + y^3 = 3axy$ at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$.
2. Find the co-ordinates of the centre of curvature at any point (x, y) of a parabola $y^2 = 4ax$. Hence find the equation of the evolute of the parabola.
3. If C_x and C_y be the chords of curvature parallel to co-ordinate axe4s at any point of the curve $y = ae^{x/a}$ prove that $\frac{1}{C_x^2} + \frac{1}{C_y^2} = \frac{1}{2aC_x}$.
4. Show that the chord of curvature through the pole of the cardioids $r = a(1+\cos\theta)$ is $\frac{4}{3}r$.
5. Show that the circle of curvature at $\left(\frac{a}{4}, \frac{a}{4}\right)$ on the curve

   $$\sqrt{x} + \sqrt{y} = \sqrt{a} \text{ is } \left(x - \frac{3a}{4}\right)^2 + \left(y - \frac{3a}{4}\right)^2 = \frac{a^2}{2}.$$

6. Show that the chord of curvature parallel to y-axis for the catenary $y = c\cosh\frac{x}{c}$ is double the ordinate of the point and the chord of curvature parallel to x-axis is $c\sinh\frac{2x}{c}$.

Remarks

7. Show that for the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, whose eparametric equations are x = a secθ, y = b tan θ, the centre of curvature at any point 'θ' is $\left(\frac{a^2+b^2}{a}\sec^3\theta, -\frac{a^2+b^2}{a}\tan^3\theta\right)$ and the equation of its evolute is $(ax)^{2/3}-(by)^{2/3} = (a^2+b^2)^{2/3}$.
8. Define the evolute of a curve and show that the evolute of
$$x^{2/3}+y^{2/3}=a^{2/3} \text{ is } (x+y)^{2/3}+(x-y)^{2/3}=2a^{2/3}$$
[Hint : The parametric equations of the curve are $x = a\cos^3\theta$, $y = a\sin^3\theta$].
9. Show that at any point on the equiangular spiral $r = ae^{\theta\cot\alpha}$, ρ = cosec α and show that the radius of curvature subtends a right angle at the pole.
10. If C_O, C_p denote the chord of curvature of the cardioide $r = a(1+\cos\theta)$ along and perpendicular to the radius vector through any point respectively. Prove that $3\left(C_O^2+C_p^2\right)=8aC_O$.

Answers

1. $\left(\frac{21a}{16},\frac{21a}{16}\right)$ 2. $\left(3x+2a, \mp\frac{2x^{3/2}}{\sqrt{a}}\right); 27ay^2 = 4(x-2a)^3$

Keywords : If ρ = radius of curvature then $\rho = \frac{dS}{d\psi}$

1. Cartesian Equation $y = f(x)$
$$\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}$$
2. Parametric Equation $x = \phi(t)$, $y = \psi(t)$
$$\rho = \frac{(x'^2+y'^2)^{\frac{3}{2}}}{x'y''-x''y'}$$
3. Polar Equation $r = f(\theta)$
$$\rho = \frac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2}$$
4. Pedal Equation $P = f(r)$
$$\rho = r.\frac{dr}{dP}$$
5. Tangential Polar Equation $P = f(\psi)$
$$\rho = P+\frac{d^2P}{d\psi^2}$$

Summary

We study curvature of the curve at a point. Then radius of curvature in different types of equations such as Cartesian, parametric, polar, pedal, tangential polar equations. Also we study centre of curvature, circle of curvature and evolute.

Remarks

CHAPTER – IV

SINGULAR POINTS

4.0 STRUCTURE

4.1 INTRODUCTION

For implicit function f(x, y) = 0

$$\rho = \frac{(f_x^2 + f_y^2)^{\frac{3}{2}}}{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2}$$

If $f_x = f_y = 0$ then ρ will be of the form $\left(\frac{0}{0}\right)$. Thus the curvature at the point has no meaning. Similarly the slope of the tangent, $-\frac{f_x}{f_y}$ for the curve f(x, y) = 0 has no meaning if $f_x = f_y = 0$. If both f_x and f_y do not vanish simultaneously at the point, then a single branch of the curve f(x, y) = 0 passes through the point. Then the point is an ordinary point. If $f_x = f_y = 0$ at the point (h, k), the point is called singular point.

4.2 OBJECTIVE : After reading this lesson, you must understand

- Multiple points
- Double points (Node, cusp, conjugate point)
- Position and nature of double points
- Species of cusps
- Point of Oscul-inflexion
- Concavity and convexity

4.3 DEFINITION

A point on a curve at which the curve exhibits an unusual or extraordinary behaviour is called a singular point.

In this chapter we shall discuss two particular types of singular points :

(i) Multiple points (ii) Points of Inflexion

4.4 MULTIPLE POINTS

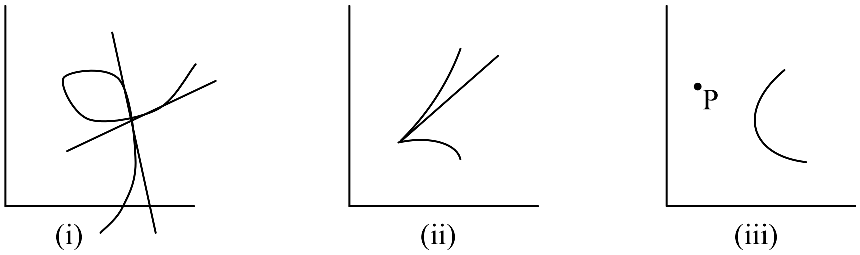
A point on the curve through which more than one branch of the curve pass, is called multiple point of the curve. It is called a Double Point, if two branches of the curve pass through it. At the double point of the curve, there are two tangents, one to each branch. These tangents may be real and distinct, real and coincident or imaginary.

4.5 TYPES OF DOUBLE POINTS

According to the nature of tangents at the double points of the curve, these double points are divide into the following three types:

Remarks

(i)Node : A double point on the curve through which two real branches of the curve pass and the tangents at which are real and distinct is called a node. [see fig. (i)].



(i) (ii) (iii)

Fig.

(ii) Cusp : A double point on the curve through which two real branches of the curve pass and the tangents at which are real and coincident is called a cusp. [see fig. (ii)].

(iii) Conjugate or Isolated Point : A point P is called a conjugate or an isolated point of the curve, if there are no real points of the curve in the neighbourhood of the point. For example, consider the curve $a^2y^2 = x^2(x^2 - a^2)$. The origin (0, 0) lies on the curve. The tangents at the origin are given by $y^2 = -x^2$ or $y = \pm\sqrt{-x}$, which are imaginary.Thus the origin is a conjugate point of the curve. [see fig. (iii)].

4.5.1 Working rule to determine the nature of the origin, if it is a double point i.e. if there are two tangents at the origin, real or imaginary.

(I) (i) If the two tangents at the origin are imaginary, then the origin is a conjugate or an isolated point of the curve.

(ii) If the two tangents at the origin are real and distinct, then the origin may be a node or a conjugate point according as the two branches of the curve through the origin are real or imaginary.

(iii) IF the two tangents at the origin are real and coincident then the origin may be a cusp or conjugate point according as the two branches of the curve through the origin are real or imaginary.

(II) In (ii) and (iii) above when we are doubtful about the nature of the double point, the following methods are useful for testing the nature of the curve at the double point (origin in the present case).

(i) If y = 0 is a cuspidal tangent at the origin, then neglect cubes and higher powers of y from the equation of the curve and solve the equation for y. If for small enough values of x; y is real, then the branches of the curve through the origin are real, otherwise imaginary.

(ii) If x = 0 is a cuspidal tangent to the curve at the origin, then neglect cubes and higher powers of x and solve the equation for x. If for small enough values of y; x is real then the branches of the curve through the origin are real, otherwise imaginary.

Example 1 : Find the nature of the origin for the curve $a^4y^2 = x^4(x^2 - a^2)$.

Solution : The given curve passes through the origin and the tangents at the origin are given by $y^2 = 0$.

i.e. $y = 0$ is a double tangent.

$\therefore$ Origin is a cusp or a conjugate point and $y = \pm\frac{x^2}{a^2}\sqrt{x^2 - a^2}$

Now near origin $x^2 - a^2 < 0$

$\therefore$ $\sqrt{x^2 - a^2}$ is imaginary and so near origin, the curve has no real branch.

Hence origin is a conjugate point.

Example 2 : Prove that the point (1, 2) is a node on the curve $y(x - 1)^2 = x(y - 2)^2$ and find the equations of the tangents to the curve at this point.

Solution : Shifting the origin to the point (1, 2) i.e. putting $x - 1 = X$, $y - 2 = Y$, we have

Remarks

$$(Y+2)X^2 = (X+1)Y^2$$
(1)

The tangents at the new origin are given by

$$2X^2 = Y^2 \qquad \Rightarrow \qquad Y = \pm\sqrt{2}X$$

$\therefore$ The new origin is either a node or a conjugate point.
Equation (1) may be written as

$$(X+1)Y^2 - X^2Y - 2X^2 = 0$$

$\therefore$ Solving for Y,

$$Y = \frac{X^2 \pm \sqrt{X^4 + 8(X+1)X^2}}{2(X+1)}$$

Now for small values of X, the quantity under the radical sign is positive. Thus y is real.

Hence the two branches of the curve that pass through the new origin are real and so the new origin is a node i.e. the point (1, 2) is a node.

The tangents at the new origin are

$$Y = \sqrt{2}\,X \qquad \text{and} \qquad y - \sqrt{2}\,X$$

i.e. $y - 2 = \sqrt{2}\,(x-1)$ and $y - 2 = -\sqrt{2}\,(x-1)$

i.e. $y = \sqrt{2}\,x - \sqrt{2} + 2$ and $y = -\sqrt{2}\,x + \sqrt{2} + 2.$

4.5.2 Working rule for finding the position and nature of double points on a curve.

1. Write the equation of the curve in the form f(x, y) = 0

2. Find the partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial^2 x}, \frac{\partial^2 f}{\partial^2 y}, \frac{\partial^2 f}{\partial x \partial y}$.

3. Solve the equations $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$.

The pair of values of x, y so obtained are the possible double points but amongst these, only those which satisfy the given equation of the curve are the required double points.

4. At each of these double point, calculate $D = \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 - \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2}$. Then if

(i) D is +ve, double point is a node or conjugate point.
(ii) D is zero, double point is a cusp or conjugate point.
(iii) D is –ve, double point is a conjugate point.

Example 1 : Find the position and the nature of the double points on the curve

$$y^2 = (x-2)^2(x-1)$$

Solution : Let $f(x, y) = (x-2)^2(x-1) = 0$
(1)

Then

$$\frac{\partial f}{\partial x} = (x-2)^2 - 2(x-2)(x-1)$$

$$\frac{\partial f}{\partial y} = -2y$$

For the double points $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$$\therefore \qquad (x-2)\,\{x - 2 - 2x + 2\} = 0$$

Remarks

i.e. $x = 2$ and $x = 0$

and $y = 0$

Thus the possible double points are (2, 0) and (0, 0). But only (2, 0) satisfies (1). Therefore (2, 0) is the only double point.

Shifting the origin at (2, 0), equation (1) transforms into

$$Y^2 = X^2(X + 1) \qquad (2)$$

The tangents at the new origin are $Y^2 = X^2$ i.e. $Y = \pm X$

Thus the new origin is either a node or a conjugate point.

Solving (2) for Y, we get $Y = \pm X\sqrt{X+1}$ which gives Y real for small values of X. Thus the branches of the curve near the new origin are real. Therefore the new origin or the point (2, 0) on (1) is a node.

Example 2 : Find the position and the nature of the double points of the curve $2(x^3 + y^3) - 3(3x^2 + y^2) + 12x - 4 = 0$

Solution : The equation of the curve is

$$f(x, y) = 2(x^3 + y^3) - 3(3x^2 + y^2) + 12x - 4 = 0 \qquad (1)$$

$\therefore$ $$\frac{\partial f}{\partial x} = 6x^2 - 18 + 12$$

and $$\frac{\partial f}{\partial y} = 6y^2 - 6y$$

For the double points $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y} = 0$

$\therefore$ $x^2 - 3x + 2 = 0 \Rightarrow (x - 1)(x - 2) = 0$

and $y(y - 1) = 0$

which gives x = 1, 2 and y = 0, 1.

Thus the possible double points are (1, 0), (1, 1), (2, 0) and (2, 1). Out of these only (1, 1) and (2, 0) satisfy (1).

Shifting the origin at (1, 1), the transformed equation is

$$2(X + 1)^3 + (Y + 1)^3 - 9(X + 1)^2 - 3(Y + 1)^2 + 12(X + 1) - 4 = 0$$

or $$2(X^3 + Y^3) + 3(Y^2 - X^2) = 0 \qquad (2)$$

Thus the tangents at the new origin are $Y^2 - X^2 = 0$ i.e., $Y = \pm X$. Thus the new origin is either a node or a conjugate point.

Neglecting Y^3 and higher powers of Y and solving for Y, we get

$$3Y^2 = 3X^2 - 2X^3$$

Near the origin for small values of X; Y is real, therefore the branches of the curve exist near the new origin. Hence the new origin i.e. the point (1, 1) of (1) is a node.

Now shifting the origin at (2, 0) i.e. putting X = X + 2 and y = Y + 0, the equation (1) transforms into

$$2(X + 2)^3 + 2Y^3 - 9(X + 2)^2 - 3Y^2 + 12(X + 2) - 4 = 0$$

or $$2(X^3 + Y^3) + 3(X^2 - Y^2) = 0 \qquad (3)$$

The tangents at the new origin are $Y = \pm X$. Therefore the new origin is a node or a conjugate point. Neglecting Y^3, equation (3) reduces to $3Y^2 = 2X^3 + 3X^2$. For small value of X; Y is real. Therefore the new origin or the point (2, 0) is a node.

Thus (1, 1) and (2, 0) are nodes for the given curve.

Remarks

Example 3 : Show that the curve $y^2 = bx\sin\frac{x}{a}$ has a node or a conjugate point at the origin according as a and b have like or unlike signs.

Solution : The equation of the curve is

$$y^2 = bx\sin\frac{x}{a} \qquad (1)$$

or

$$y^2 = bx\left[\frac{x}{a} - \frac{1}{3!}.\frac{x^3}{a^3} + \frac{1}{5!}.\frac{x^5}{a^5} - ....\right]$$

$$= b\left[\frac{x^2}{a} - \frac{x^4}{6a^3} + \frac{x^6}{120a^5} - ....\right] \qquad (2)$$

The curve passes through the origin, as there is no constant term in (2). Tangents at the origin are given by equating to zero the lowest degree terms in (2).

$\therefore$

$$y^2 = \frac{b}{a}x^2$$

or

$$y = \pm\sqrt{\frac{b}{a}}x \qquad (3)$$

Since there are two tangents at the origin, therefore origin is a double point.

Case I : When a and b are of like signs, then $\frac{b}{a}$ is positive and so (3) gives two real and distinct tangents. Therefore origin is a node or a conjugate point.

Also from (2), the behaviour of the expression on R.H.S. is same as that of the first term for small values of x, near the origin.

In other words, (2) behaves as $y^2 = \frac{b}{a}x^2$.

Since $\frac{b}{a}$ is positive, so y^2 is +ve for small values of x, (+ve or −ve). Thus y is real in the neighbourhood of origin and so origin is a node.

Case II : If a and b are of unlike signs, then $\frac{b}{a}$ is negative and from (3), the two tangents are imaginary.

Thus origin is a conjugate point.

Example 4 : Find the position and the nature of the double points of the curve, $x^3 + x^2 + y^2 - x - 4y + 3 = 0$.

Solution : The equation of the curve is

$$f(x, y) = x^3 + x^2 + y^2 - x - 4y + 3 = 0 \qquad (1)$$

$\therefore$

$$\frac{\partial f}{\partial x} = 3x^2 + 2x - 1 \quad \text{and} \quad \frac{\partial f}{\partial y} = 2y - 4$$

For the double points, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y} = 0$

Remarks

$\therefore \qquad 3x^2 + 2x - 1 = 0 \Rightarrow \qquad (3x - 1)(x + 1) = 0$

and $\qquad 2(y - 2) = 0$

which gives, $\quad x = \dfrac{1}{3}, -1$ and $y = 2$

Thus the possible double points are $\left(\dfrac{1}{3}, 2\right)$ and $(-1, 2)$. Of these only $(-1, 2)$ satisfies (1).

Shifting the origin at $(-1, 2)$ the transformed equation is

$$(X - 1)^3 + (X - 1)^2 + (Y + 2)^2 - (X - 1) - 4(Y + 2) + 3 = 0$$

or

$$X^3 - 2X^2 + Y^2 = 0 \qquad (2)$$

$\therefore$ The tangents at the new origin are

$$Y^2 - 2X^2 = 0 \quad \Rightarrow \quad Y = \pm\sqrt{2}X$$

Thus the new origin is either a node or a conjugate point.

Solving (2) for Y, we get $\quad Y = \pm X\sqrt{2 - X}$

Near the origin for small values of X (+ve or –ve), Y is real, therefore two real branches of the curve exist near the new origin. Hence $(-1, 2)$ is a node.

Exercise 4.1

1. Examine the nature of the origin on the curve $x^4 + y^4 - 4xy = 0$.

Find the position and nature of double points on the following curves :

2. $x^4 + 4ax^3 - 2ay^3 + 4a^2x^2 + 3a^2y^2 - a^4 = 0$
3. $x^4 + 4ax^3 + 4a^2x^2 - b^2y^2 - 2b^3y - a^4 - b^4 = 0$
4. $x^4 + y^3 + 2x^2 + 3y^2 = 0$
5. $(y - x)^2 + x^6 = 0$
6. $(2y + x + 1)^2 = 4(1 - x)^5$.
7. Examine the curve $x^3 + 2x^2 + 2xy - y^2 + 5x - 2y = 0$ for a singular point and show that it is a cusp.
8. Prove that the only singular point on the curve $(y - b)^2 = (x - a)^3$ is a cusp and find its coordinates.
9. Find the equations of the tangents at the multiple points of the curve $(y - 2)^2 = x(x - 1)^2$.
10. Find the equations of the tangents at the multiple points of the curve $(x - 2)^2 = y(y - 1)^2$.

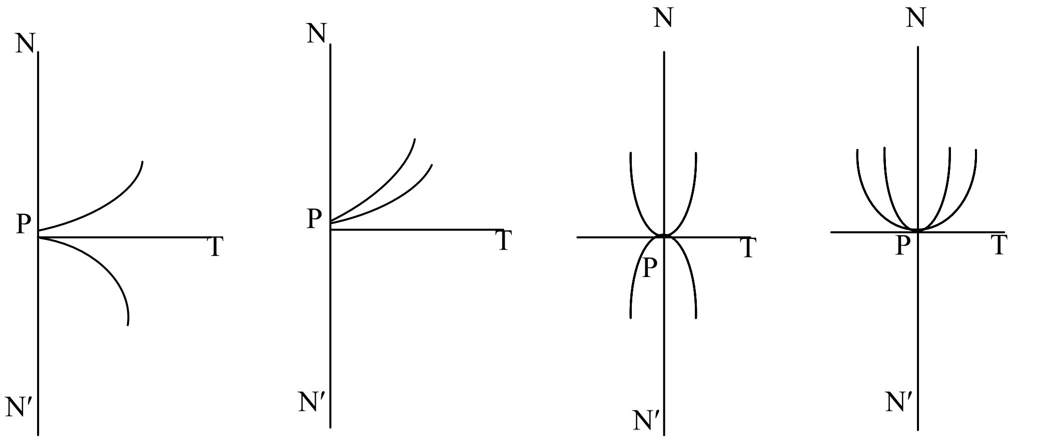
Answers

1.	Node	**2.**	Node at point (0, a), (–a, a) and (–a, 0)
3.	Conjugate point at (–a, –b)	**4.**	Conjugate point at (0, 0)
5.	(0, 0) is a conjugate point	**6.**	Cusp at (1, –1)
8.	(a, b) 9. $y - x - 1 = 0$, $y + x - 3 = 0$	**10.**	$x - y - 1 = 0$, $x + y - 3 = 0$

Remarks

4.6 SPECIES OF CUSPS

When the double points of a curve is a cusp, then the two tangents at this point are coincident and therefore the normal to the two branches of the curve is also common. A cusp is called a single cusp or a double cusp according as the two branches of the curve lie entirely on one side or on both the sides of the normal. Again the cusp, single or double, is said to be of first species or of second species

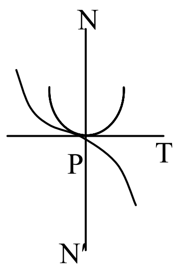


(i) single cusp (ii) single cusp (iii) Double cusp
(iv) Double cusp
of first species of second species of first species
of second species

according as the two branches of the curve lie on the opposite or the same side of the coincident tangents. In figures PT is the common tangent and NPN′ is the common normal to the two branches of the curve where P is the cusp.

Point of oscul-inflexion : A double cusp with change of species on the two sides of the common normal is called a point of oscul-inflexion.

A cusp of first species is also called a keratoid cusp (i.e. a cusp like horns) and a cusp of the second species is also called a ramphoid cusp (i.e., a cusp like a beak).



Point of oscul-inflexion

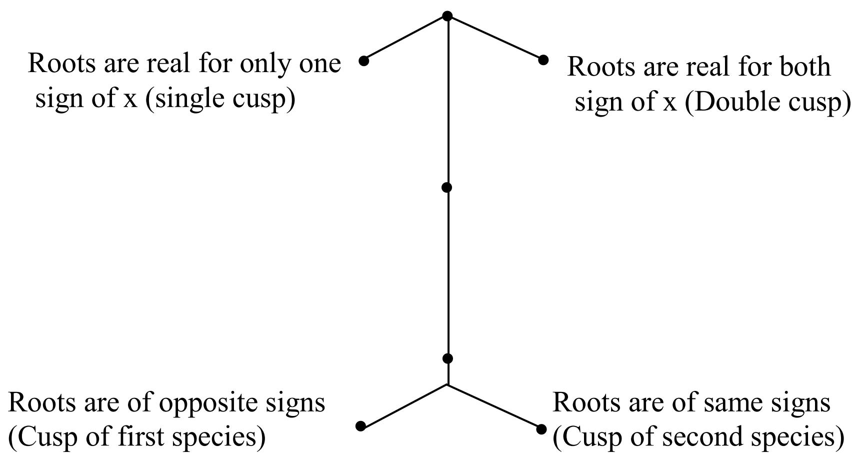
4.6.1 Working rule to the nature of a cusp at the origin

Case I : When the cuspidal tangents are $y^2 = 0$, i.e. x-axis.

Solve the equation of the curve for y, neglecting cubes and higher powers of y. The branches of the curve can be discussed by the following tree diagram.

Remarks

Nature of cusp at the origin
When x-axis is a cuspidal tanget



Case II : When the cuspidal tangents are $x^2 = 0$, i.e. y-axis.

Solve the equation of the curve for x, neglecting cubes and higher powers of x and discuss the branches of curve, as in case I.

Case III : When the cuspidal tangent is of the form $ax + by = 0$, then put $p = ax + by$. Eliminate y between this equation and the equation of the curve so as to get a relation between p and x. Solving this resultant equation for p, neglecting cubes and higher powers of p, we discuss the nature of the branches of the curve for small values of x.

Example 1 : Show that the curve $(2x + y)^2 - 6xy(2x + y) - 7x^3 = 0$ has a single cusp of first species at the origin.

Solution : The given curve is $(2x + y)^2 - 6xy(2x + y) - 7x^3 = 0$ (1)

Obviously the curve passes through the origin. Equating the lowest degree terms of (1) to zero i.e. $(2x + y)^2 = 0$ are the tangents at the origin. Thus the origin is a cusp or a conjugate point and $2x + y = 0$ is a cuspidal tangent.

put $p = 2x + y$ [case III]

or $y = p - 2x$ (2)

Eliminating y from (1) and (2), we get

$$p^2 - 6px(p - 2x) - 7x^3 = 0$$

or $$p^2 - 6p^2x + 12px^2 - 7x^3 = 0$$

or $$(1 - 6x)p^2 + 12x^2p - 7x^3 = 0$$

which is a quadratic equation in p

Solving, $$p = \frac{-12x^2 \pm \sqrt{144x^4 + 28x^3(1-6x)}}{1-6x}$$

$$= \frac{-12x^2 \pm \sqrt{28x^3 - 28x^4}}{1-6x}$$

Since for small values of $x > 0$, $x^4 < x^3$; therefore $28x^3 - 24x^4 > 0$ and therefore p is real. Thus the origin is a single cusp. Again neglecting $-24x^4$, we get

Remarks

$$p = \frac{-12x^2 \pm \sqrt{28x^3}}{1-6x}$$

and we have $28x^3 > 144x^4$ if $x < \frac{28}{144} = \frac{7}{36}$

i.e. the two values of p are of opposite sign. Therefore the cusp is of first species. Hence the curve has a single cusp of first species at the origin.

Example 2 : Show that the curve $x^2(2a - y) = y^3$ has a single cusp of first species at the origin.

Solution : The equation of the curve is $x^2(2a - y) = y^3$ (1)

The tangents at the origin are given by equating to zero the lowest degree terms in (1)

i.e. $2ax^2 = 0 \quad \Rightarrow \quad x^2 = 0,$

which are real and coincident.

Thus the origin is a cusp or conjugate point.

From (1),
$$x = \pm y\sqrt{\frac{y}{2a-y}} \qquad (2)$$

When y is small and negative, x is real. Thus origin is a cusp.

Also from (2), x is real only if y is small and +ve (i.e. x is real for only one sign of y). Therefore the cusp is a single cusp.

Again for any small positive value of y, the two values of x are of opposite sign. Therefore the cusp is of first species.

Hence origin is a single cusp of first species.

Example 3 : Show that the curve $(x + y)^3 - \sqrt{2}\,(y - x + 2)^2 = 0$ has a single cusp of the first species at the point (1, −1).

Solution : the given equation of the curve is

$$(x + y)^3 - \sqrt{2}\,(y - x + 2)^2 = 0 \qquad (1)$$

Shifting the origin to the point (1, −1), by putting x = X + 1, y = Y + 1, we have from equation (1),

$$(X + Y)^3 - \sqrt{2}\,(Y - X)^2 = 0 \qquad (2)$$

Equating to zero the lowest degree terms in (2), the tangents at the new origin are $(Y - X)^2 = 0$, which are real and coincident.

Thus the new origin is either a cusp or a conjugate point.

From (2), neglecting Y^3, we have

$$X^3 + 3X^2Y + 3XY^2 - \sqrt{2}\,(Y^2 - 2XY + X^2) = 0$$

or
$$Y^2(3X - \sqrt{2}\,) + X(2\sqrt{2} + 3X)Y - X^2(\sqrt{2} - X) = 0$$

which is a quadratic in Y.

$\therefore$

$$Y = \frac{-X(2\sqrt{2}+3X) \pm \sqrt{X^2(2\sqrt{2}+3X)^2 + 4X^2(\sqrt{2}-X)(3X-\sqrt{2})}}{2(3X-\sqrt{2})}$$

$$= \frac{-X(2\sqrt{2}+3X) \pm X\sqrt{28\sqrt{2}X - 3X^2}}{2(3X-\sqrt{2})}$$

For every small positive values of X, $\sqrt{28\sqrt{2}X - 3X^2}$ is positive and thus Y is real.

Remarks

Hence two real branches of the curve pass through the new origin.
Thus the new origin is a cusp.

Put $p = Y - X \quad \Rightarrow \quad Y = X + p$ (3)

Eliminating Y between (2) and (3), we get

$$(p + 2X)^3 - \sqrt{2}\, p^2 = 0$$

or $$p^3 + 6Xp^2 + 12X^2p + 8X^3 - \sqrt{2}\, p^2 = 0$$

Then $$(6X - \sqrt{2})p^2 + 12X^2p + 8X^3 = 0.$$ [Neglecting p^3]

Solving, $$p = \frac{-12X^2 \pm \sqrt{144X^4 - 32X^3(6X - \sqrt{2})}}{2(6X - \sqrt{2})}$$

$$= \frac{-6X^2 \pm \sqrt{8\sqrt{2}X^3 - 12X^4}}{6X - \sqrt{2}}$$

$$= \frac{-6X^2 \pm \sqrt{8\sqrt{2}X^3}}{6X - \sqrt{2}}$$

[Neglecting $-12X^4$]

Thus p is real only when X is positive (i.e., of one sign)
Hence the new origin is a single cusp.

Now $\sqrt{8\sqrt{2}X^3}$ i.e., $\sqrt{8\sqrt{2}X^{3/2}} > 6X^2$, for small values of X as $X^{3/2} > X^2$, when X is small.
Thus the value of p have opposite sign which shows that cusp is of first species.
Hence the new origin or point (1, –1) is a single cusp of first species

.**Example 4 :** Show that $y^5 - ay^3x - ay^2x + a^2x^2 = 0$ has a point of oscul-inflexion at the origin.
Solution : The equation of the curve is

$$x^5 - ay^3x - ay^2x + a^2x^2 = 0$$

The equations of the tangents at the origin are

$$x^2 = 0 \quad \text{i.e.} \quad x = 0, x = 0$$

Thus the origin is a cusp or a conjugate point.
Writing the equation as a quadratic in x, we have

$$a^2x^2 - (ay^3 + ay^2)x + y^5 = 4$$

Solving for x, $$x = \frac{(ay^3 + ay^2) \pm \sqrt{(ay^3 + ay^2)^2 - 4a^2y^5}}{2a^2}$$

$$= \frac{(ay^3 + ay^2) \pm (ay^3 - ay^2)}{2a^2} = \frac{y^3}{a}, \frac{y^2}{a}.$$

If y > 0, x > 0 for both the branches.
If y < 0, x < 0 for one branch and x > 0 for the other branch.
Hence the origin is a point of inflexion.

Example 5 : Show that the curve $x^5 - 2x^3y - 4x^2y + + 8y^2 = 0$ has a point of oscul-inflexion at the origin.\
Solution : The equation of the curve is $x^5 - 2x^3y - 4x^2y + + 8y^2 = 0$
The equations of the tangents at the origin are given by

$$y^2 = 0 \quad \text{i.e.} \quad y = 0, y = 0$$

Thus the origin is a cusp or a conjugate point
Writing the equation as a quadratic in y, we have

$$8y^2 - (2x^3 + 4x^2)y + x^5 = 0$$

Remarks

Solving for y, we have

$$y = \frac{(2x^3 + 4x^2) \pm \sqrt{(2x^3 + 4x^2)^2 - 32x^5}}{16}$$

$$= \frac{(2x^3 + 4x^2) \pm (2x^3 - 4x^2)}{16} = \frac{x^3}{4}, \frac{x^2}{2}.$$

If x > 0, for both the branches, y > 0

If x , 0, for one branch y > 0 and for the other branch y < 0.

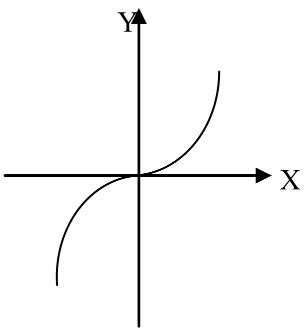
Hence origin is a point of oscul-inflexion.

Exercise 4.2

1. Show that the following curves have a single cusp of first species at the origin :
   (i) $y^3 = x^3 + ax^2$ (ii) $x^2(x + y) - y^2 = 0$
2. Show that the curve $y^2 - 2x^2y + x^4 - x^5 = 0$ have a single cusp of second species at the origin.
3. Show that the curve $y^3 = (x - a)^2(2x - a)$ has a single cusp of first species at (a, 0).
4. Shat the following curves have a point of oscul-inflexion at the origin :
   (i) $x^5 + 16x^2y - 64y^2 = 0$ (ii) $x^7 + 2x^4 + 2x^3y + 2xy + y^2 + x^2 = 0$.
5. Show that the curve $x^5 - ax^3y - a^2x^2y + a^3y^2 = 0$ has a point of oscul-inflexion at the origin.

4.7 CONCAVITY AND CONVEXITY

Definitions : Let a function f be continuous on a closed interval [a, b] and differentiable in open interval (a, b). Then



(i) f is **concave upwards** on [a, b] if, throughout (a, b), the graph of f lies above the tangent lines to f.

(ii) f is **concave downwards** on [a, b] if, throughout (a, b), the graph of lies below the tangent lines to f.

4.5.1 Conditions for the concavity or convexity of the curve in the Upward Direction

Concave upward $\frac{d^2y}{dx^2} > 0$ or Concave downward $\frac{d^2y}{dx^2} < 0$

4.8 POINT OF INFLEXION

Definition : A point on a curve, at which the curve changes from concavity to convexity or vice-versa, is called a point of inflexion of the curve.

4.8.1 Test for Point of Inflexion

Remarks

If f has an inflexion point at the number c in (a, b), then either $f''(c) = 0$ or f'' does not exist at c. The converse is not necessarily true.

1. Find all numbers at which $f''(x) = 0$
2. Use the test for concavity.
3. If the concavity changes, there is inflexion point; otherwise, there is not.

Thus a point is a point of inflexion, if $\frac{d^2y}{dx^2} = 0$ at this point and $\frac{d^2y}{dx^2}$ changes sign in passing through this point, i.e., $\frac{d^3y}{dx^3} \neq 0$ at this point.

Example 1: Find the range of values of x for which the curve $y = 3x^5 - 40x^3 + 3x - 20$ is concave upwards or downwards. Also find the points of inflexion on the curve.

Solution : The given curve is $y = 3x^5 - 40x^3 + 3x - 20$ (1)

$$\therefore \quad \frac{dy}{dx} = 15x^4 - 120x^2 + 3$$

$$\frac{d^2y}{dx^2} = 60x^3 - 240x = 60x(x^2 - 4) \tag{2}$$

and
$$\frac{d^3y}{dx^3} = 180x^2 - 240 \tag{3}$$

The curve (1) is concave upwards if $\frac{d^2y}{dx^2} > 0$

i.e., if $x(x^2 - 4) > 0$

or if $x(x - 2)(x + 2) > 0$

which is possible when either $x \in (-2, 0)$ or $x > 2$ (4)

and the curve is concave downwards if $\frac{d^2y}{dx^2} < 0$

i.e. if $x(x - 2)(x + 2) < 0$

i.e. when either $x < -2$ or $x \in (0, 2)$ (5)

Equations (4) and (5) give the range of values for the concavity or convexity of the curve in the upward direction.

The points of inflexion are given by $\frac{d^2y}{dx^2} = 0$

Which gives $x = 0$, $x = -2$, $x = 2$.

For all these values of x, $\frac{d^3y}{dx^3} \neq 0$. Therefore these are the abscissa of the points of inflexion of the curve.

For $x = 0$, $y = -20$.

For $x = -2$, $y = 3(-2)^5 - 40(-2)^3 + 3(-2) - 20$

$= -96 + 320 - 6 - 20 = 198$

Remarks

For $x = 2$, $y = 3(2)^5 - 40(2)^3 + 3(2) - 20$

$= 96 - 320 + 6 - 20 = -238$

Hence the points of inflexion are (0, –20), (–2, –198) and (2, –238).

Example 2 : Find the points of inflexion on the curve $x = a(2\theta - \sin\theta)$, $y = a(2 - \cos\theta)$.

Solution : Differentiating the given equation w.r.t. θ, we have

$$\frac{dx}{d\theta} = a(2 - \cos\theta) \text{ and } \frac{dy}{d\theta} = a\sin\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\sin\theta}{2 - \cos\theta}$$

and
$$\frac{d^2y}{dx^2} = \frac{(2 - \cos\theta)\cos\theta - \sin\theta(\sin\theta)}{(2 - \cos\theta)^2}.\frac{d\theta}{dx}$$

$$= \frac{2\cos\theta - \cos^2\theta - \sin^2\theta}{(2 - \cos\theta)^2}.\frac{1}{a(2 - \cos\theta)} = \frac{2\cos\theta - 1}{a(2 - \cos\theta)^3}$$

For the points of inflexion, $\frac{d^2y}{dx^2} = 0$

i.e., $2\cos\theta - 1 = 0 \quad \Rightarrow \quad \cos\theta = \frac{1}{2} = \cos\frac{\pi}{3}$

$\therefore \quad \theta = 2n\pi \pm \frac{\pi}{3}$, where n is any integer.

Obviously $\frac{d^2y}{dx^2}$ changes sign when θ passes through each of the values given above. Hence each value of θ corresponds to the point of inflexion on the curve. The co-ordinates of the points of inflexion are given by

$$x = a\left[4n\pi \pm \frac{2\pi}{3} \mp \frac{\sqrt{3}}{2}\right], y = \frac{3a}{2}.$$

Example 3 : Find the points of inflexion of the curve $y = (x - 2)^6(x - 3)^5$.

Solution : The given curve is $y = (x - 2)^6(x - 3)^5$

$$\therefore \quad \frac{dy}{dx} = (x - 2)^6.5(x - 3)^4 + 6(x - 2)^5(x - 3)^5$$

$$= (x - 2)^5(x - 3)^4[5(x - 2) + 6(x - 3)]$$

$$= (x - 2)^5(x - 3)^4(11x - 28)$$

$$\frac{d^2y}{dx^2} = (x - 2)^5(x - 3)^4.11 + (x - 2)^5(11x - 28).4(x - 3)^3 + (x - 3)^4(11x - 28).5(x - 2)^4$$

$$= (x - 2)^4(x - 3)^3[11(x - 2)(x - 3) + 4(x - 2)(11x - 28) + 5(x - 3)(11x - 28)]$$

$$= (x - 2)^4(x - 3)^3[110x^2 - 560x + 710]$$

$$= 10(x - 2)^4(x - 3)^3(11x^2 - 56x + 71)$$

Remarks

$$= 10(x-2)4(x-3)^3\left[x - \frac{28+\sqrt{3}}{11}\right]\left[x - \frac{28-\sqrt{3}}{11}\right]$$

At the points of inflexion $\frac{d^2y}{dx^2} = 0$

$\therefore$ $$x = 2, 3, \frac{28 \pm \sqrt{3}}{11}$$

At x = 2, $\frac{d^2y}{dx^2}$ does not change sign as the corresponding factor $(x-2)^4$ remains +ve whether x < 2 or x > 2.

$\therefore$ x = 2 does not correspond to the point of inflexion.

Also $\frac{d^2y}{dx^2}$ changes sign at x = 3 and $x = \frac{28 \pm \sqrt{3}}{11}$

Thus these correspond to the abscissa of the points of inflexion of the given curve.

Example 4 : Find the points of inflexion on the curve $x^2y = a^2(x - y)$.

Solution : The given curve is $$x^2y = a^2(x-y)$$

or $$y = \frac{a^2x}{x^2 + a^2} \qquad (1)$$

$\therefore$ $$\frac{dy}{dx} = a^2\left[\frac{(x^2+a^2) - x.2x}{(x^2+a^2)^2}\right] = \frac{a^2(a^2 - x^2)}{(x^2+a^2)^2}$$

and $$\frac{d^2y}{dx^2} = a^2\left[\frac{(x^2+a^2).(-2x) - (a^2 - x^2).2(x^2+a^2).2x}{(x^2+a^2)^4}\right]$$

$$= \frac{2a^2x(x^2 - 3a^2)}{(x^2+a^2)^3}$$

At the points of inflexion, $$\frac{d^2y}{dx^2} = 0$$

$\Rightarrow$ $$2a^2 x(x^2 - 3a^2) = 0$$

$\Rightarrow$ $$x = 0, \quad \pm\sqrt{3}a .$$

At x = 0, $\frac{d^2y}{dx^2}$ changes sign from +ve to –ve and so there is a point of inflexion.

Also $\frac{d^2y}{dx^2}$ changes sign from –ve to +ve at $x = \sqrt{3}a$, and thus there is a point of inflexion.

At $x = -\sqrt{3}a$, $\frac{d^2y}{dx^2}$ changes sign from +ve to –ve and so there is a point of inflexion.

Hence the points of inflexion are (0, 0), $\left(\pm\sqrt{3}a, \pm\frac{\sqrt{3}a}{4}\right)$.

Remarks

Exercise 4.3

1. Find the points of inflexion on the following curves

   (i) $r(\theta^2 - 1) = a\theta^2$ (ii) $r^2\theta = a^2$.

2. Show that points of inflexion on the curve $r = b\theta^n$ are given by $r = b[-n(n+1)]^{n/2}$.
3. Find the range of values of x for which the curve $y = x^4 - 6x^3 + 12x^2 + 5x + 7$ is concave or convex upwards. Determine also the points of inflexion.
4. Find the points of inflexion for the following curves :

   (i) $y(a^2 + x^2) = x^3$ (ii) $y = 3x^4 - 4x^3 + 1$

5. Find the points of inflexion of the curve $x = (y - 1)(y - 2)(y - 3)$.
6. Show that the points of inflexion of the curve $y^2 = (x - a)^2(x - b)$ lies on the line $3x + a = 4b$.
7. Show that the abscissa of the point of inflexion on the curve $x = a - b\cos\theta$, $y = a\theta - b\sin\theta$ is $\dfrac{a^2 - b^2}{a}$.
8. Show that the abscissa of the points of inflexion of the curve $y^2 = f(x)$, satisfy the equation $[f'(x)]^2 = 2f(x)\, f''(x)$.
9. Prove that the curve $y = \log x$ is everywhere convex upwards.
10. Show that every point in which the sine curve $y = c\sin\dfrac{x}{c}$ meets the axis of x, is a point of inflexion.

Answers

1. *(i)* $\left(\dfrac{3a}{2}, \pm\sqrt{3}\right)$ *(ii)* $\left(\sqrt{2a}, \dfrac{1}{2}\right)$

3. The curve is concave upwards in $(-\infty, 1)$ and $(2, \infty)$ and concave downwards in (1,2). The points of inflexion are (1, 19) and (2, 33).
4. (i) $(0,0)\left(\pm\sqrt{3a}, \pm\dfrac{3\sqrt{3a}}{4}\right)$ (ii) $(0,1)$ and $\left(\dfrac{2}{3}, \dfrac{11}{27}\right)$
5. (0, 2).

Keywords : Node, Cusp, Conjugate point, point of Oscul-inflexion, point of inflexion.

Summary

We discussed two particular types of singular points (i) multiple points (ii) points of inflexion. A point on the curve through which more than one branch of the curve pass is called multiple point. If two branches of the curve passes through a point then it is called double point. Three types of double points (Node, cusp, conjugate point). If the curve passes through origin then to find tangents at (0, 0), equate the lowest degree term in x & y = 0. We studied the species of cusps, point of Oscul-inflexion, point of inflexion and test for point of inflexion.

Remarks

CHAPTER – V

CURVE TRACING

5.0 STRUCTURE

5.1 INTRODUCTION

The object of curve tracing is to find the approximate shape of a curve without plotting a large number of points. For this, we shall use the information from the previous chapters like tangents and normals, maxima and minima, asymptotes, concavity etc.

5.2 OBJECTIVE : After reading this lesson, you must be able to understand

- Tracing of Cartesian curve
- Tracing of Parametric equations
- Tracing of polar curves

5.3 TRACING OF CARTESIAN CURVES

The following steps will help us in drawing the rough sketch of the Cartesian curves.

1. Symmetry : Notice if the curve is symmetrical about any line, by applying the following rules :

(i) *Symmetry about the x-axis :* The curve is symmetrical about the x-axis if the equation remains unchanged on replacing y by –y or if the equation contains only even powers of y. (See fig. 5.1)

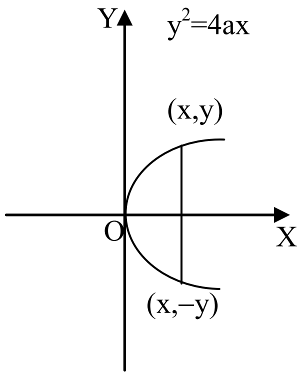


Fig. 5.1

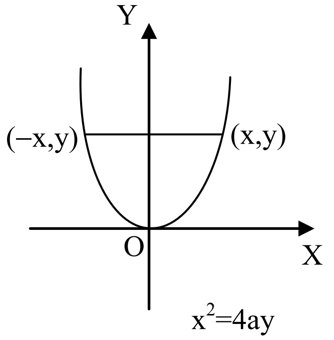


Fig. 5.2

(ii) *Symmetry about the y-axis :* The curve is symmetrical about the y-axis if the equation remains unchanged when x is replaced by –x or if the equation contains only even powers of x. (see fig. 5.2).

(iii) *Symmetry about the origin :* The curve is symmetrical in the opposite quadrants if the equation remains unchanged when x and y are replaced by –x and –y. (see fig. 5.3)

Remarks

(iv) *Symmetry about the line $y = x$* : The curve is symmetrical about the line $y = x$ if the equation is

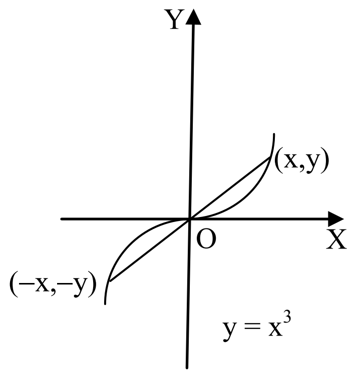


Fig. 5.3

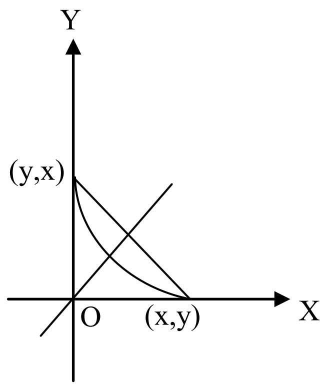


Fig. 5.4

unaltered when x is changed to y and y is changed to x. (see fig. 5.4).

(v) *Symmetry about the line $y = -x$* : The curve is symmetrical about the line $y = -x$ if the equation remains unchanged when x is changed to $-y$ and y is changed to $-x$. (see fig. 5.5).

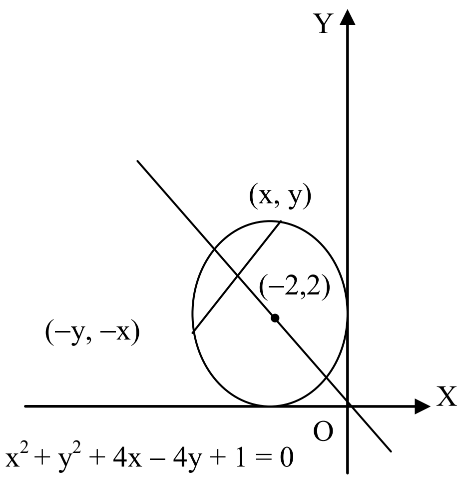


Fig. 5.5

2. Origin : If the constant term is missing from the equation of the curve, then the curve passes through the origin.

If the curve passes through the origin, then write down the tangent or tangents at the origin by equating to zero the lowest terms. If the origin is a double point, then find its nature as under.

Remarks

Nature of origin
If it is a double point

Node
(if tangents are real
and distinct)

Cusp
(if tangents are real
and coincident)

Conjugate point
(if tangents are imaginary)

3. Asymptotes : Find the asymptotes to the curve by the following methods :

(i) *Asymptotes parallel to x-axis(y-axis) :* The asymptotes parallel to x-axis(y-axis) are obtained by equating to zero the co-efficients of highest powers of x(y) present in the equation of the curve.

(ii) *Oblique asymptotes :* Find $\phi_n(m)$ by putting $x = 1$, $y = m$ in the highest degree terms of x and y.

Solve $\phi_n(m) = 0$ to get the values of m (i.e., slopes of the asymptotes).

Find $\phi_{n-1}(m)$ by putting $x = 1$, $y = m$ in the next lowest degree terms of x and y and similarly find $\phi_{n-2}(m)$.

If values of m are real and distinct, find c from the equation

$$c = -\frac{\phi_{n-1}(m)}{\phi'_n(m)}$$

If two values of m are real and equal, find c from the equation

$$\frac{c^2}{2!}\phi''_n(m) + c\phi'_n(m) + \phi_{n-2}(m) = 0$$

Use the value of m and c in $y = mx + c$ to get the oblique asymptotes.

4. Points of Intersection

(i) The curve meets x axis where $y = 0$. Putting $y = 0$, if we get real values of x, then the curve meets x-axis. Similarly we examine the curve if it meets y-axis where $x=0$.

(ii) Shift the origin to the points one by one and find the tangents at the new origin and discuss the nature of this new origin also as in (i).

(iii) If the curve is symmetrical about $y = \pm x$, then find the points of intersection of the curve with these lines also.

5. Region : The region to which the curve is bounded in the quadrants can be found as given below:

(i) Solve the equation of the curve for y. Find those values of x for which y is real. Then the curve lies between these values of x for e.g., in $9ay^2 = (x - 2a)(x - 5a)^2$, y^2 becomes –ve when $x < 2a$ i.e., y is imaginary for $x < 2a$. So no portion of the curve lies to the left of the line $x = 2a$.

(ii) Reject those values of x and y which make L.H.S. and R.H.S. of the equation of opposite sign.

(iii) Put $x = 0$ and observe how y will vary as x increases and then tends to ∞, paying particular attention to those values of x for which $y = 0$, or $y \to \infty$.

(iv) Observe the variations of y as x decreases from 0 to $-\infty$. But if the curve is symmetrical about y-axis, consider only the +ve values of x and trace the curve for –ve values of x by symmetry.

6. Special points : Sometimes it becomes necessary to find few additional points. For example (i) the points were the curve has maxima and minima of the given function (ii) the region of concavity or convexity and the points of inflexion.

Remarks

Find the points where $\frac{dy}{dx}$ is

- Zero (Tangents are parallel to x-axis)
- ∞ (Tangents are parallel to y-axis)
- + ve (curve is increasing)
- – ve (curve is decreasing)

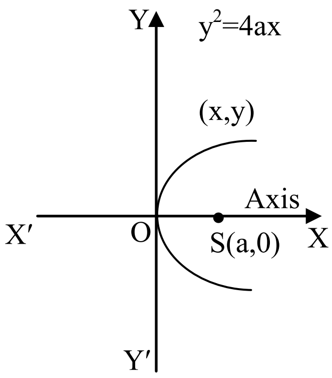
Find the points where $\frac{d^2y}{dx^2}$ is

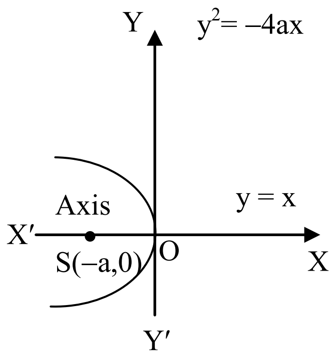
- + ve (Curve is concave upward)
- – ve (Curve is concave downward)

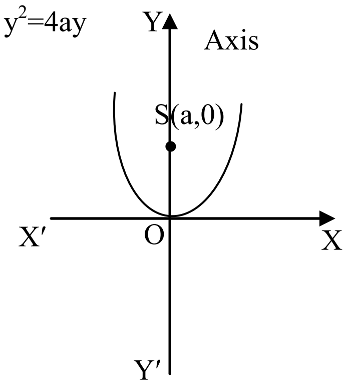
Find the points where $\frac{d^2y}{dx^2} = 0$ and $\frac{d^3y}{dx^3} \neq 0$ i.e., points where $\frac{d^2y}{dx^2}$ changes sign while passing through it. This gives the point of inflexion.

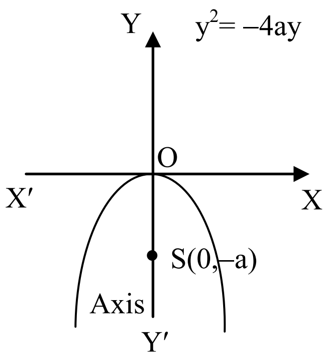
5.4 PARABOLA

As the students are already familiar with these curves, we give below just its graphs in all the four cases :







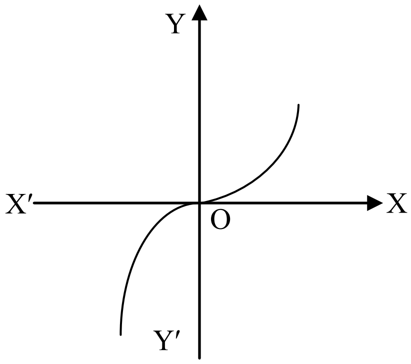


Remarks

5.4.1 Cubical Parabola : $y = x^3$

Symmetry : The curve is symmetrical in opposite quadrants, as changing x into –x and y into –y, the equation of the curve does not change.

Origin : The curve passes through the origin. The tangent at the origin is y = 0 i.e., x-axis is a tangent at the origin.



Region : When x < 0, then y , 0 and when x . 0, then y > 0. Therefore the curve lies in the I and III quadrant only.

Asymptotes : The curve has no asymptote

Special points $\frac{dy}{dx} = 3x^2$

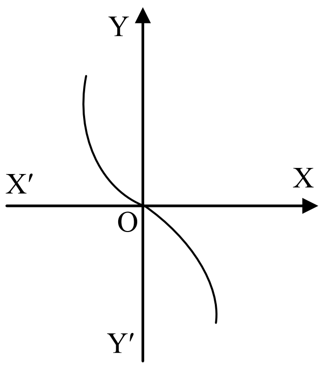
and $\frac{d^2y}{dx^2} = 6x$

In the I quadrant, when x > 0, $\frac{d^2y}{dx^2} > 0$, therefore the curve is concave upwards. In the III quadrant, when x < 0, then $\frac{d^2y}{dx^2} < 0$. Then the curve is concave downwards.

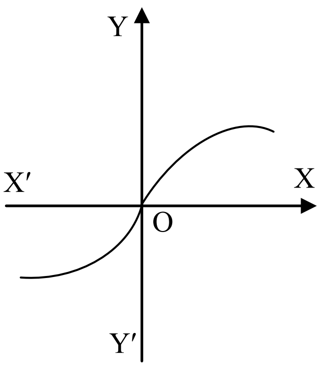
$\frac{d^2y}{dx^2} = 0$, when x = 0 and thus y = 0 and because $\frac{d^2y}{dx^2}$ changes sign while passing through the origin, therefore the origin is a point of inflexion. Lastly when $x \to \infty$, then y also $\to \infty$, but more rapidly. And when $x \to -\infty$, again more rapidly. Therefore for the larger values of x, the inclination of the curve is towards y-axis. Hence the shape of the curve is as shown in above figure.

Three similar forms are drawn as under

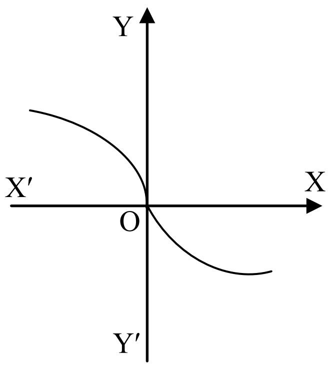
$y = -x^3$



$x = y^3$



$x = -y^3$



5.4.2 Semi Cubical Parabola : Here the equation is of the form $y^2 = x^3$.

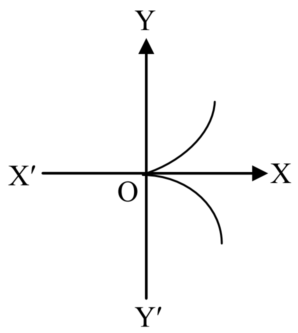
1. Symmetry : Because of only even powers of y, the curve is symmetrical about x-axis.

2.Origin : The curve passes through the origin. The tangents at the origin are $y^2 = 0$ i.e., x-axis is a cuspidal tangent.

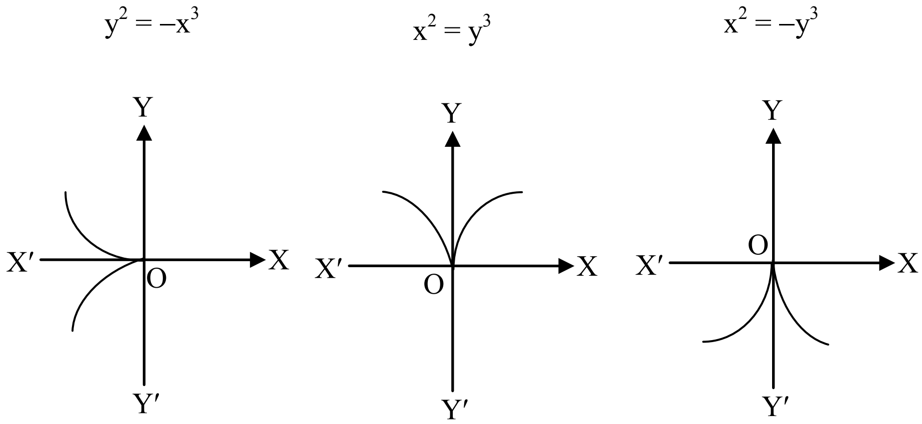
3. Region : When x < 0, then y is imaginary. Therefore no part of the curve lies to the left of y-axis. Again when x > 0, there are two equal and opposite values of y. Thus the origin is a single cusp of first species. Again when $x \to \infty$, y also $\to \pm\infty$, but a bit more rapidly. Therefore for larger values of x, the inclination of the curve is towards y-axis.

Remarks

4. Asymptotes : The curve has no asymptote. Hence the shape of the curve is shown in figure below



Three similar forms are as under



Example 1 : Trace the curve $y^2 = (x - 1)(x - 2)(x - 3)$.

Solution : The equation of the curve is $y^2 = (x - 1)(x - 2)(x - 3)$.

1. Symmetry : Because of only even powers of y, the curve is symmetrical about x-axis.

2. Origin : The curve does not pass through the origin.

3. Asymptote : The curve has no asymptote.

4. Points of intersection with the co-ordinate axes : The curve does not meet y-axis, for when $x = 0$, then $y^2 = -6$ which gives imaginary values of y. Again it meets x-axis, where $y = 0$, which gives $x = 1$, $x = 2$ and $x = 3$.

Shifting the origin at (1, 0), the equation transform into $Y^2 = X(X - 1)(X - 2)$

The tangent at the new origin is $X = 0$ i.e., $x = 1$ is a tangent at (1, 0). Similarly $x = 2$ is a tangent at (2, 0) and $x = 3$ is a tangent at (3, 0).

5. Region : (i) When $x < 1$, y is imaginary (ii) When $1 < x < 2$, y is real (iii) When $2 < x < 3$, y is imaginary. (iv) When $x > 3$, y is real.

Therefore the curve lies between $x = 1$ and $x = 2$ and for $x > 3$. For larger values of x i.e., when $x \to \infty$, then y also $\to \infty$, but more rapidly.

6. Special Points :

$$y^2 = (x - 1)(x - 2)(x - 3)$$

or

$$y^2 = x^3 - 6x^2 + 11x - 6$$

Differentiating w.r.t. x, we have

$$2y\frac{dy}{dx} = 3x^2 - 12x + 11$$

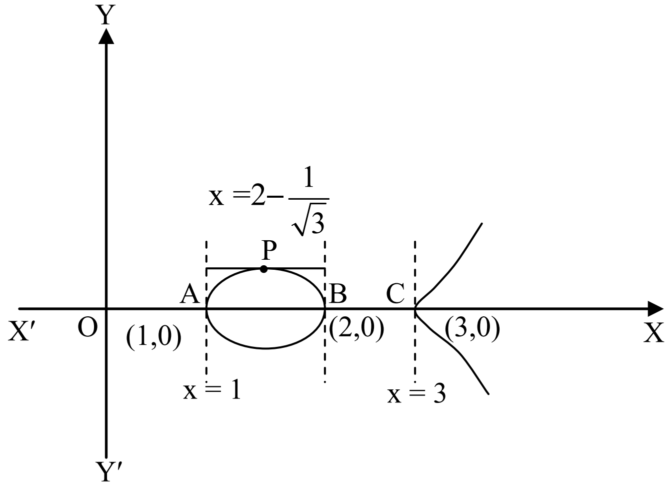
$$\therefore \quad \frac{dy}{dx} = 0 \Rightarrow \quad 3x^2 - 12x + 11 = 0$$

Remarks

$\therefore$ $$x = \frac{12 \pm \sqrt{144-132}}{6} = \frac{12 \pm 2\sqrt{3}}{6} = 2 \pm \frac{1}{\sqrt{3}}$$

We reject the +ve sign, as no part of the curve lies between x = 2 and x = 3. Therefore the tangent to the curve is parallel to x-axis between x = 1 and x = 2 i.e., the curve has a turning point at x $= 2 - \dfrac{1}{\sqrt{3}}$

Hence the shape of the figure is shown as under



Example 2 : Trace the curve $y^2(a^2 + x^2) = x^2(a^2 - x^2)$

or $x^2(x^2 + y^2) = a^2(x^2 - y^2)$.

Solution : The given equation of the curve is $x^2(x^2 + y^2) = a^2(x^2 - y^2)$

1. Symmetry : Because of only even powers of x and y, the curve is symmetrical about both the axes.

2. Origin : The curve passes through the origin. The tangents at the origin are given by $y^2 = 0$ or $y = \pm x$ which are real and distinct. Therefore the origin is a node.

3. Asymptotes : The equation of the curve is of 4^{th} degree in x and y. The term containing x^4 is present, whereas the terms containing y^4 and y^3 are absent. The co-efficient of y^2 is $a^2 + x^2$. Therefore asymptotes parallel to y-axis, given by $x^2 + a^2 = 0$ are imaginary i.e., there is no asymptote parallel to either of the axes. For oblique asymptotes $\phi_n(m) = 1 + m^2$, which again does not give any real value of m. Therefore there is no oblique asymptote either.

4. Points of intersection with co-ordinate axes : The curve meets x-axis where y = 0, which gives x = 0 and x = ± a. The point (0, 0) i.e., the origin has already been discussed. Shifting the origin at (a, 0) i.e., putting x = X + a, y = Y + 0, the equation transforms into

$$Y^2[a^2 + (X + a)^2] = (X + a)^2(X + 2a)(-X).$$

The tangent at the new origin is X = 0 or x – a = 0 i.e., x = a is a tangent at the point (a, 0). Similarly because because of symmetry about both the axes x = –a is a tangent at the point (–a, 0).

5. Region : Writing the equation in the form

$$y^2 = \frac{x^2(a^2 - x^2)}{a^2 + x^2}$$

i.e., $$y = \pm x\sqrt{\frac{a^2 - x^2}{a^2 + x^2}},$$

Remarks

we see that y is real for $x^2 \le a^2$ or $|x| \le a$ or $-a \le x \le a$ i.e., the whole of the curve lies between the lines $x = -a$ and $x = a$.

6. Special Points : Differentiating,

$$y^2 = \frac{a^2x^2 - x^4}{a^2 + x^2}$$

$$2y\frac{dy}{dx} = \frac{(a^2+x^2)(2a^2x - 4x^3) - (a^2x^2 - x^4)2x}{(a^2+x^2)^2}$$

$\therefore$ $\frac{dy}{dx} = 0 \Rightarrow \quad 2x[a^2 + x^2)(a^2 - 2x^2) - (a^2x^2 - x^4)] = 0$

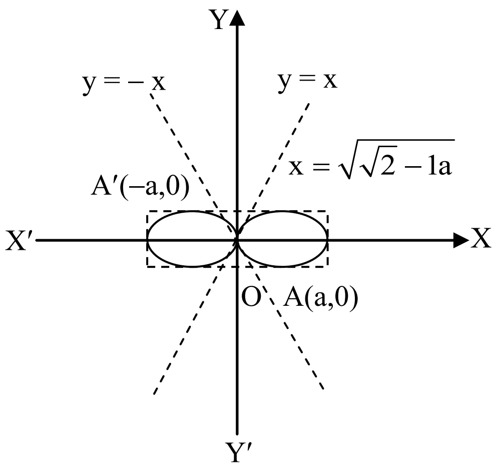
$\Rightarrow -2x[x^4 + 2a^2x^2 - a^4] = 0$

$\therefore$ either $x = 0$ or $x^2 = \frac{-2a^2 \pm \sqrt{4a^4 + 4a^4}}{2} = (-1 \pm \sqrt{2})a^2.$

Rejecting –ve sign, we get $x^2 = \left(\sqrt{2} - 1\right)a^2$

or $x = \pm\sqrt{\sqrt{2} - 1a}$

Thus the tangent is parallel to x-axis between x = 0 and x = a. Hence the form of the curve is as shown in the figure drawn below :



Example 3 : Trace the curve $x^2y^2 = (a + y)^2(a^2 - y^2)$.

Solution : The given equation of the curve is

$$x^2y^2 = (a + y)^2 (a^2 - y^2$$

or $$x^2 = (a+y)^2 \frac{a^2 - y^2}{y^2}$$

1. Symmetry : Because of even powers of x only, the curve is symmetrical about y-axis.

2. Origin : The curve does not pass through the origin.

3. Asymptotes : The equation of the curve is of 4th degree in x and y. The terms containing x^4 and x^3 are absent. The coefficient of x^2 is y^2. Therefore the asymptote parallel to x-axis is y = 0 i.e., the x-axis itself. There is no other asymptote to the curve.

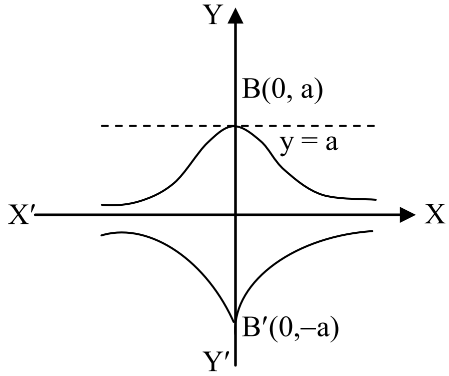
Remarks

4. Points of intersection with co-ordinate axes : The curve does not meet x-axis. It meets y-axis, where $x = 0$. Therefore $y = -a$, $y = a$.

Shifting the origin at $(0, a)$, the equation of the curve transforms into

$$X^2(Y + a)^2 = (Y + 2a)^3 (-Y).$$

Thus the equation of the tangent at the new origin is $Y = 0$ or $y = a$ is a tangent at $(0, a)$. Again shifting the origin at $(0, -a)$, the equation transforms into $X^2(Y - a)^2 = Y^3(2a - Y)$. Therefore the tangents at the new origin are given by $X^2 = 0$ i.e. y-axis is a cuspidal tangent at $(0, -a)$.



5. Region : The whole of the curve lies between $y = -a$ and $y = a$.

Hence the shape of the curve is as shown in the above figure.

Example 4 : Trace the curve $x^2y^2 = x^2 - a^2$.

Solution : The equation of the curve is $x^2y^2 = x^2 - a^2$.

1. Symmetry : Because of only even powers of both x and y, the curve is symmetrical about both the axes.

2. Origin : The curve does not pass through the origin.

3. Asymptotes : It is a 4^{th} degree equation in x and y. The coefficient ofx^2 equated to zero gives $y^2 - 1 = 0$ i.e. $y = \pm 1$ are two asymptotes parallel to x-axis. The coefficient of y^2 is x^2. But $x^2 = 0$ cannot give asymptotes as the curve does not lie between $x = -a$ and $x = a$.

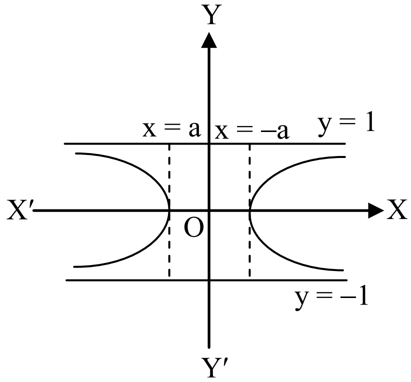
4. Points of intersection with the co-ordinate axes : The curve does not meet y-axis. It meets x-axis where $y = 0$, therefore $x = \pm a$.

Shifting the origin at $(a, 0)$, the equation is transformed into $(X + a)^2Y^2 = (X + 2a)X$.

The tangent at the new origin is $X = 0$

i.e., the line $x = a$ is a tangent at the origin.

Similarly the line $x = -a$ is a tangent at $(-a, 0)$.



5. Region : From $y = \pm\frac{\sqrt{x^2 - a^2}}{x}$, we see that y is real only if $|x| \geq a$. Again from $x^2(1 - y^2) = a^2$, we have $x = \pm\frac{a}{\sqrt{1 - y^2}}$, therefore x is real, if $|y| < 1$.

Thus the whole curve lies between the lines $y = \pm 1$ and beyond the lines $x = \pm a$.
Hence the shape of the curve is as shown in the above figure.

Example 5 : Trace the curve $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1.$

Solution : The given curve is $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$

1. Symmetry : If we write the equation as

Remarks

$$\left[\left(\frac{x}{a}\right)^{1/3}\right]^2 + \left[\left(\frac{y}{b}\right)^{1/3}\right]^2 = 1;$$

We see that equation contains only even powers of both x and y. Therefore the symmetry is about both the axes.

2. Origin : The curve does not pass through the origin.

3. Asymptotes : The curve has no asymptote as it is a closed curve.

4. Points of intersection with co-ordinate axes : The curve meets x-axis where y = 0, i.e., $\left(\frac{x}{a}\right)^{1/3} = 1$ or $\left(\frac{x}{a}\right)^{2} = 1$ (cubing both sides) or x = ± a. Similarly it meets y-axis where y = ± b.

5. Region : From the equation of the curve

$$\left(\frac{y}{b}\right)^{2/3} = 1 - \left(\frac{x}{a}\right)^{2/3}$$

or $$\left(\frac{y}{b}\right)^{1/3} = \pm\sqrt{1 - \left(\frac{x}{a}\right)^{2/3}}$$

or $$\frac{y}{b} = \pm\frac{(a^{2/3} - x^{2/3})^{3/2}}{a}$$

or $$y = \pm\frac{b}{a}(a^{2/3} - x^{2/3})^{3/2}$$

which gives y real only when | x \ ≤ a. Thus the whole curve lies between the lines x = ± a. Similarly, writing x explicitly in terms of y, we see that the whole curve lies between the lines y = ± b. Thus the whole curve lies between the rectangle formed by the lines x = ± a and y = ± b.

6. Special Points : Differentiating the given equation w.r.t. x, we get

$$\frac{2}{3}\left(\frac{x}{a}\right)^{-1/3}\frac{1}{a} + \frac{2}{3}\left(\frac{y}{b}\right)^{-1/3}\frac{1}{b}\frac{dy}{dx} = 0$$

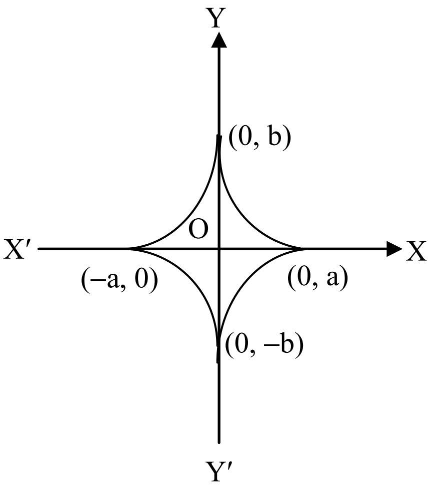
or $$\frac{dy}{dx} = \frac{b^{2/3}}{a^{2/3}}.\frac{y^{1/3}}{x^{1/3}}$$

∴ $$\frac{dy}{dx} = 0$$

⇒ $y = 0$

⇒ $$\left(\frac{x}{a}\right)^{2/3} = 1$$

⇒ $x = \pm a$



∴ y = 0 i.e., x-axis is tangent to the curve at 9a, 0) and (–a, 0).

Since the curve is symmetric about x-axis, therefore x-axis is a cuspidal tangent at both these points.

Similarly y-axis i.e., x = 0 is a cuspidal tangent at the points (0, b) and (0, –b).

Find $\frac{d^2y}{dx^2}$. We observe that $\frac{d^2y}{dx^2} > 0$ when both x and y are +ve. Thus the curve is concave upwards in the first quadrant.

Remarks

Again because the curve is symmetrical about both the axes, hence the shape of the curve as shown in the figure.

Remark : 1. Putting b = a, the equation of the curve becomes $x^{2/3} + y^{2/3} = a^{2/3}$ which is called four cusped asteroid and the shape of the curve is almost the same with b=a.

2. The parametric equations of the curve

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$$

are $x = a\cos^3 t$ and $y = b\sin^3 t$

and parametric equations of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ are

$x = a\cos^3 t$ and $y = a\sin^3 t$.

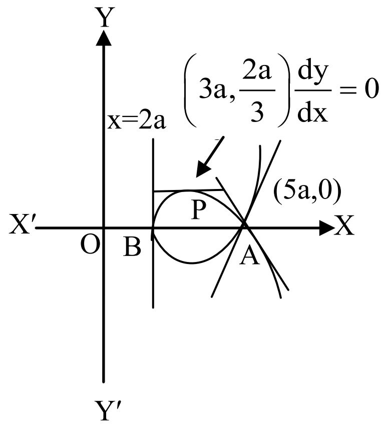
Exercise 5.1

Trace the following curves :

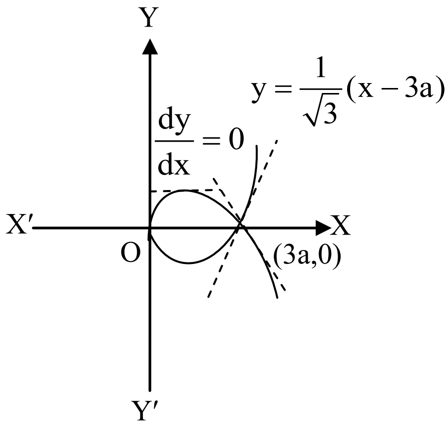
1. $9ay^2 = (x - 2a)(x - 5a)^2$
2. $x^2(y^2 + a^2) + y^2(y^2 - a^2) = 0$ or $y^2(x^2 + y^2) + a^2(x^2 - y^2) = 0$
3. $9ay^2 = x(x - 3a)^2$ 4. $y^2(2a - x) = x^3$ 5. $x^2y^2 = a^2(y^2 - x^2)$
6. $x^3 + y^3 = 3axy$ 7. $y(x^2 + 4a^2) = 8a^3$ 8. $y = x + \frac{1}{x}$
9. $x^4 + y^4 = 4a^2xy$

Answers

1.



3.



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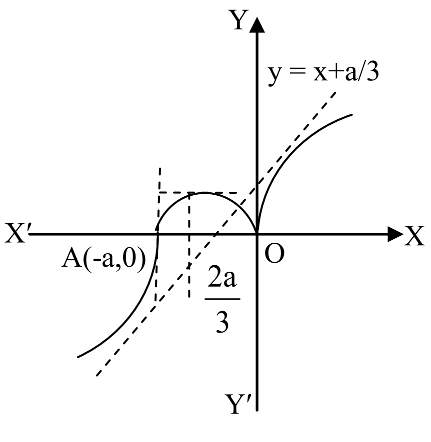

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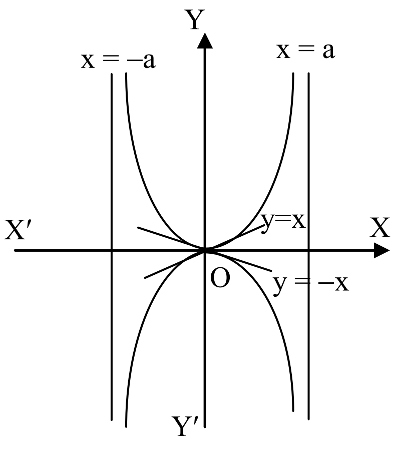
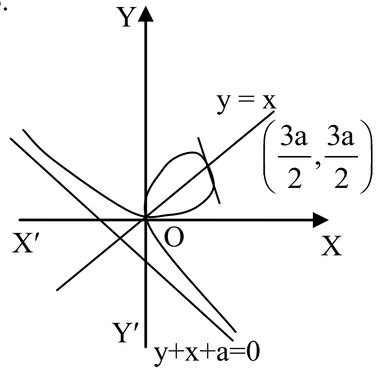


Remarks

6.



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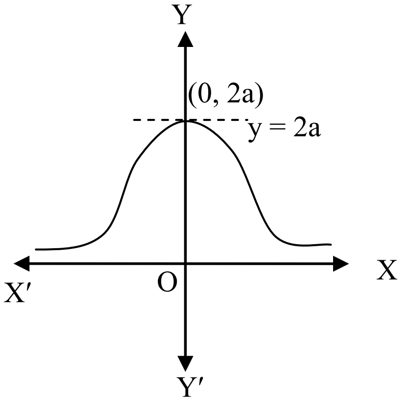
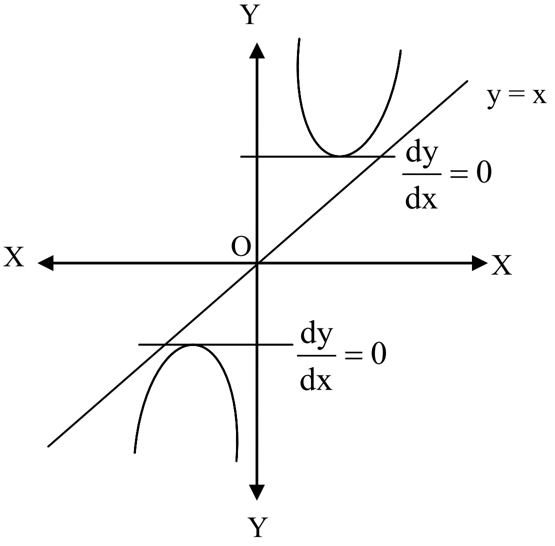


8.



5.5 PARAMETRIC EQUATION

1. Symmetry : (i) If by changing t to –t or t to π– t, x changes to –x and y remains unchanged, then the curve is symmetrical about y-axis.

e.g., $x = a \cos t$, $y = b \sin t$ (Ellipse)

(ii) Similarly if by changing t to –t or t to π– t, y changes to –y and x remains unchanged, then the curve is symmetrical about x-axis.

e.g., $x = at^2$, $y = 2at$ (Parabola)

(iii) If by changing t to –t, x changes into –x and y changes into –y, then the symmetry is in opposite quadrants.

e.g., $x = ct$, $y = \frac{c}{t}$ (Rectangular hyperbola)

2. Origin : If a common real value of t satisfies both f(t) = 0 and φ(t) = 0, then the curve passes through the origin.

Remarks

3. Asymptotes : First find asymptotes parallel to the co-ordinate axes. To get the asymptotes parallel to x-axis, find the values $b_1, b_2, \ldots$ to which y tends as $x \to +\infty$ or $-\infty$, then $y = b_1$, $y = b_2$ = are asymptotes parallel to x-axis.

Similarly find the values $a_1, a_2, \ldots$ to which x tends as $y \to +\infty$ or $-\infty$, then $x = a_1$, $x = a_2$ are asymptotes parallel to y-axis.

Oblique asymptote : For some value of t say $t \to t_0$, both x and y tend to $\pm\infty$ and $\lim_{t \to t_0} \frac{y}{x} = m$ and $\lim_{t \to t_0} (y - mx) = c$, then y = mx + c is an oblique asymptote.

4. Points of intersection with co-ordinate axes : The curve meets x-axis, where $y = \phi(t) = 0$ and it meets y-axis where $x = f(t) = 0$.

5. Region : If possible, find the greatest and least values of x and y for some real values of t, which shall give lines parallel to axes of co-ordinates, between which the curve lies or does not lie.

e.g., for the curve $x = a\cos^3\theta$, $y = b\sin^3\theta$, the greatest value of x is a and the least value is $-a$.

(since $\cos\theta$ and $\sin\theta$ both lie between -1 and 1, so the curve lies between the lines $x = \pm a$. Similarly, the curve lies completely between the lines $y = \pm b$.)

6. Special Points :

Find

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

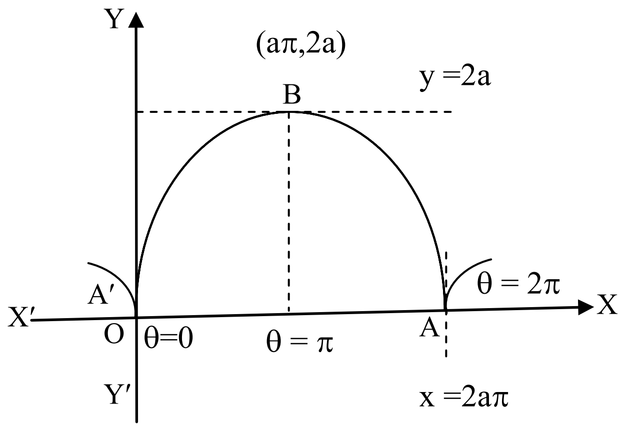
which is the slop of the tangent at $[f(t), \phi(t)]$. Find the points where $\frac{dy}{dx} = 0$ or ∞ i.e., the points where the tangent is parallel to x-axis or to y-axis. Again, if need be, find $\frac{d^2y}{dx^2}$ and discuss the concavity or convexity and the points of inflexion on the curve.

5.6 CYCLOID

A cycloid is a curve described by a point fixed in the plane of the circle (the generating circle) when the circle rolls without sliding along some straight line (the directrix). We shall consider the locus of the point, which lies on the rim of the circle.

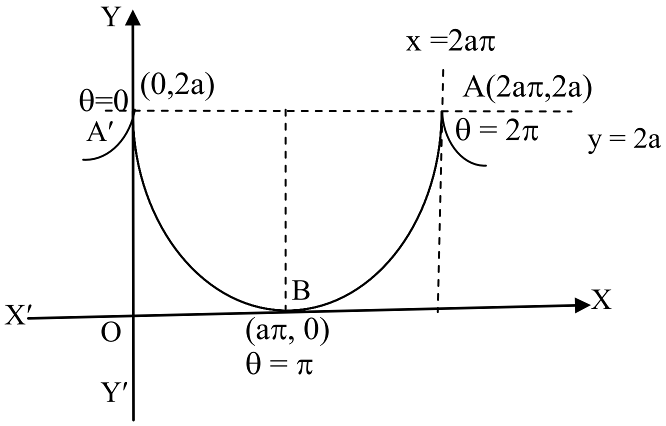
The cycloid can have the following four types of equations with their respective figures.

(i) $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, $0 \le \theta \le 2\pi$

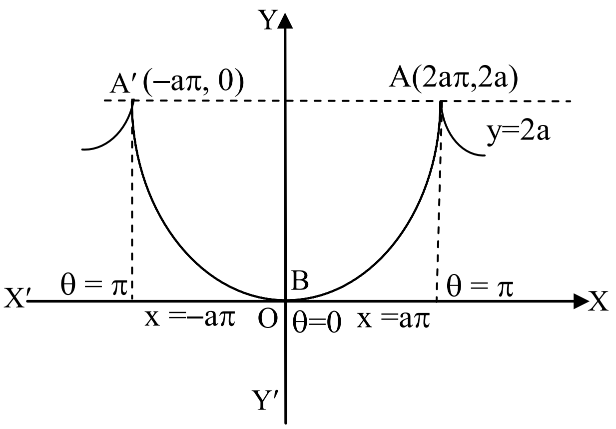


Remarks

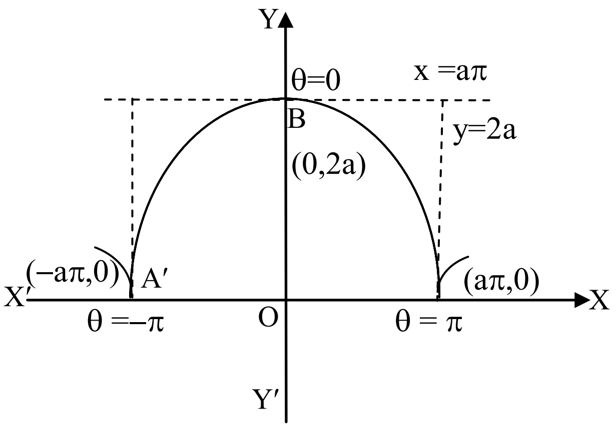
(ii) $x = a(\theta - \sin\theta),\ y = a(1 + \cos\theta),\ 0 \le \theta \le 2\pi$



(iii) $x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta),\ -\pi \le \theta \le \pi$



(iv) $x = a(\theta + \sin\theta),\ y = a(1 + \cos\theta),\ -\pi \le \theta \le \pi$



Example 1 : Trace the curve $x = t^2,\ y = t - \frac{1}{3}t^3$.

Solution : Eliminating t, we get $9y^2 = x(x - 3)^3$.

Remarks

1. Symmetry : As the equation contains only even powers of y, therefore the curve is symmetrical about x-axis.
2. Origin : The curve passes through the origin. The tangent at the origin is $x = 0$ i.e., y-axis is the tangent at the origin.
3. Asymptotes : The curve has no asymptotes.
4. Points of intersection with the axes : The curve does not meet y-axis except at the origin. The curve meet x-axis at $x = 0$ and $x = 3$. When $x = 0$, $y = 0$ and when $x = 3$, $y = 0$. We shift the origin at (3, 0) i.e., put $x = X + 3$, $y = Y + 0$.

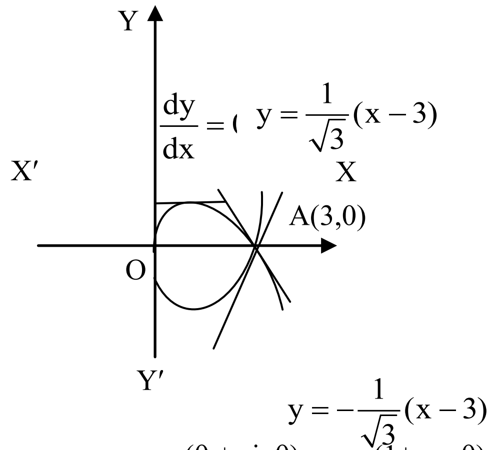
Then the equation transforms into $9Y^2 = (X + 3)X^2$

The tangents at the new origin are $9Y^2 = 3X^2$

or $$Y = \pm\frac{1}{\sqrt{3}}X$$

Hence the new origin or the point (3, 0) is a node.
5. Region : When $x < 0$, y is imaginary. Therefore no part of the curve lies to the left of the y-axis. When $x > 3$ i.e. the curve is inclined towards y-axis.

Hence the shape of the curve is as shown in the below figure :



Example 2 : Trace the curve x = = a(θ + sinθ), y = a(1+ cosθ), $-\pi \le \theta \le \pi$.
Solution : We trace this curve when θ varies from $-\pi$ to π
1. Symmetry : When θ is changed in $-\theta$, x changes into $-x$ and y remains unchanged. The curve is, therefore, symmetrical about y-axis.
2. Origin : There is no common value of θ for which x and y both are zero. Thus the curve does not passes through the origin.
3. Asymptotes : The curve has no asymptotes.
4. Points of intersection with axes : The curve meets x-axis where $y = 0$ i.e., $1+ \cos\theta = 0$ or $\theta = \pi$ and $\theta = -\pi$. The corresponding values of x are $a\pi$ and $-a\pi$.

It meets y-axis where $x = 0$, which gives $\theta = 0$ and then $y = 2a$.
5. Region : Since $|\cos\theta| \le 1$, $0 \le y \le 2a$.
Thus the whole of the curve lies between $y = 0$ and $y = 2a$.
6. Special Points : We have

$$\frac{dx}{d\theta} = a(1+\cos\theta) \text{ and } \frac{dy}{d\theta} = -a\sin\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = -\frac{a\sin\theta}{a(1+\cos\theta)}$$

Remarks

$$= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = -\tan\frac{\theta}{2}$$

$\therefore$ $\quad \frac{dy}{dx} = 0 \quad \Rightarrow \quad \tan\frac{\theta}{2} = 0$

$\Rightarrow \theta = 0$ i.e., tangent is parallel to x-axis where $\theta = 0$, which gives $x = 0$, $y = 2a$ i.e., at $(0, 2a)$.

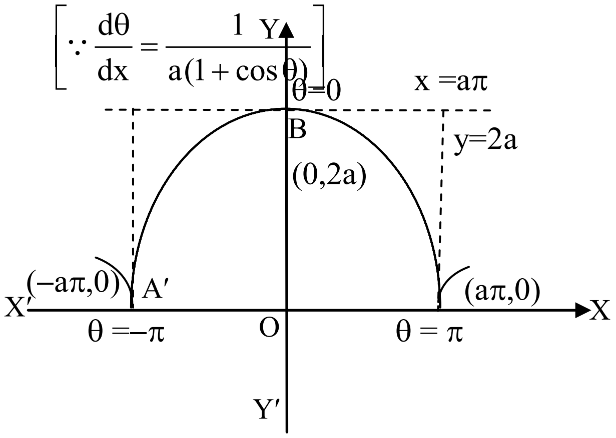
$\frac{dy}{dx} = \pm\infty \quad \Rightarrow \quad \theta = -\pi$ or π

which gives the corresponding points $(-a\pi, 0)$ and $(a\pi, 0)$.
Again

$$\frac{d^2y}{dx^2} = -\sec^2\frac{\theta}{2}.\frac{1}{2}\frac{d\theta}{dx} = -\frac{1}{2}\sec^4\frac{\theta}{2} \qquad \left[\because \frac{d\theta}{dx} = \frac{1}{a(1+\cos\theta)}\right]$$

$\therefore$ $\quad \frac{d^2y}{dx^2} < 0$ for all values of θ.

Thus the curve is concave downwards.
When $\theta < -\pi$ or $\theta > \pi$, the point traces similar curves.
The shape of the curve is as shown below :



Exercise 5.2

Trace the following curves :

1. $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, $0 \le \theta \le 2\pi$.

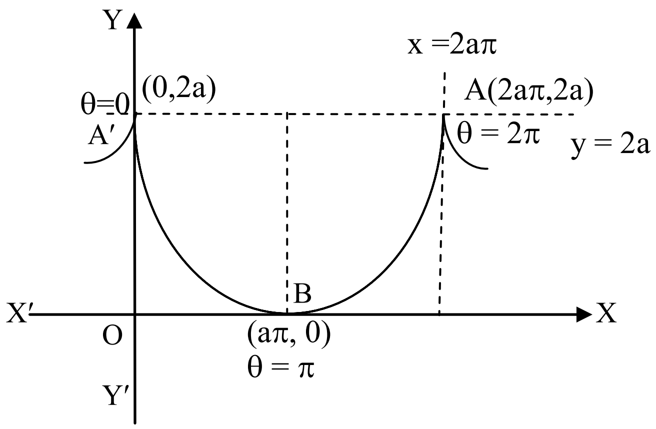
2. $x = \frac{a(t+t^3)}{1+t^4}, y = \frac{a(t-t^3)}{1+t^4}$

3. $x = \frac{3at}{1+t^3}, y = \frac{3at^2}{1+t^3}$

4. $x = a\cos^3 t$, $y = a\sin^3 t$

Answers

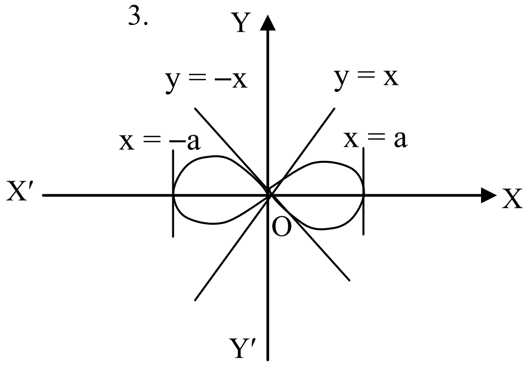
1.

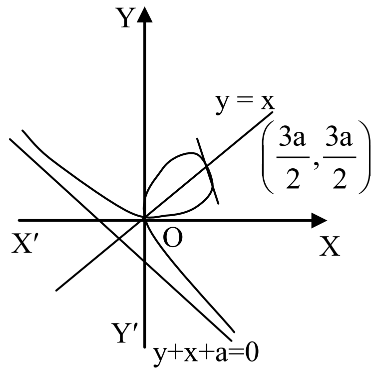


Remarks

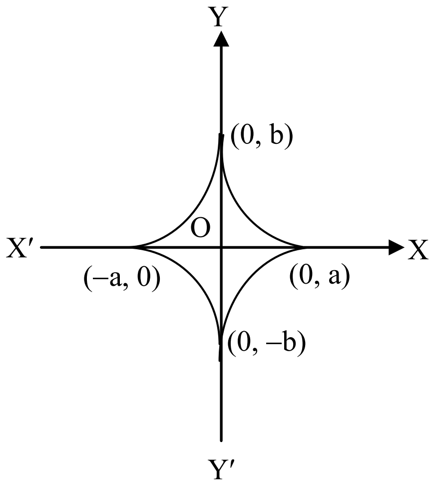
2.

3.





4.



5.7 TRACING OF POLAR CURVES

The following steps may be taken while tracing the curves in polar co-ordinates :

1. Symmetry :

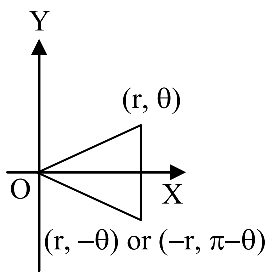
(i) The curve is symmetrical about the x-axis (initial line, $\theta = 0$) if the equation is unchanged when θ is replaced by $-\theta$, or the pair (r, θ) by the pair $(-r, \pi-\theta)$.

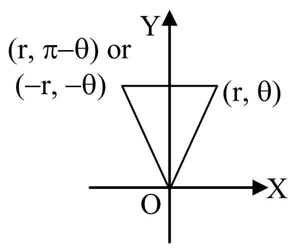
e.g. $r \cos \theta = a \sin^2\theta$.

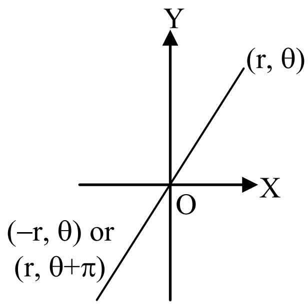
(ii) The curve is symmetrical about the y-axis (the ray $\theta = \pi/2$) if the equation is unchanged when θ is replaced by $\pi-\theta$, or by the pair $(-r, -\theta)$.

e.g. $r\theta = a$.

(iii) The curve is symmetrical about the pole if the equation is unchanged when r is replaced by $-r$, or θ is replaced by $\theta + \pi$.







Remarks

2. Origin (or the Pole) : Check whether the curve passes through the pole or not. For this put r = 0. If we get some real value of θ, then the curve passes through the pole. If we cannot find real value of θ for which r = 0, the curve does not pass through the pole. i.e. if on putting r = 0, we get $\theta = \theta_1, \theta_2, \ldots$ Where $\theta_1, \theta_2, \ldots$ are all real numbers, then the curve passes through the pole and $\theta = \theta_1$, $\theta = \theta_2 \ldots$ Are the tangents at the pole. But if on putting r = 0, we do not get any real value of θ, then the curve does not pass through the pole.

e.g. For the curve $r^2 = a^2 \cos 2\theta$, for r = 0, we get

$\cos 2\theta = 0$ or $\theta = \pm\frac{\pi}{4}$ (real values of θ)

So the curve passes through the pole and $\theta = \pm\frac{\pi}{4}$ are the tangents at the pole.

Again for the curve $r^2\theta = a^2$, putting r = 0, we get $\theta = \infty$.

Thus the curve does not pass through the pole.

3. Asymptotes : If $r \to \infty$ as $\theta \to \theta_1$ (finite value), then there is an asymptote and we find it as follows :

If α is a root of the equation f(θ) = 0, then $r \sin(\theta - \alpha) = \frac{1}{f'(\alpha)}$ is an asymptote of the curve $\frac{1}{r} = f(\theta)$.

4. Points of Intersection : Find the points of intersection with the lines θ = 0, $\theta = \frac{\pi}{6}, \theta = \frac{\pi}{4}, \theta = \frac{\pi}{3}$ and so on, or we make the table of the values of r corresponding to some suitable values of θ (especially for those values of θ for which the curve is symmetrical).

5. Region : Solve the equation for r and consider how r varies as θ increases from 0 (or some convenient value θ_1) to + ∞ and also as θ diminishes from 0 (or θ_1) to –∞.

Find the regions in which the curve does not lie. This can be checked as follows :

(i) If r is imaginary, say for $\alpha < \theta < \beta$, then no portion of the curve lies between the rays θ = α and θ = β.

(ii) If r_{max} is a for all real values of θ, then the whole of the curve lies within a circle of radius a, and if r_{min} is b, the whole of the curve lies outside the circle of radius b.

6. Special Points : Find ϕ(i.e., angle between the tangent and the radius vector) at P(r, θ) using the relation $\tan\phi = r\frac{d\theta}{dr}$

Find the points where ϕ = 0 or $\frac{\pi}{2}$.

Also if $\frac{dr}{d\theta}$ is +ve, then r increases as θ increases and if $\frac{dr}{d\theta}$ is –ve, then r decreases as θ decreases.

Note : Sometimes it is helpful to change the equations from the catesian to polar co-ordinates or from polar to Cartesian co-=ordinates and then trace the curve accordingly. For example, the curve

$$r = \frac{3a\sin\theta\cos\theta}{\cos^3\theta + \sin^3\theta}$$

Is the same as $x^3 + y^3 = 3axy$ which is the folium of Descarte's.

The curve $r \cos\theta = a \sin^2\theta$ is the same as $y^2(a - x) = x^3$, which is Cissoid.

Example 1 : Trace the curve r = a cos 2θ.

Solution : The given curve is r = a cos 2θ.

1. Symmetry : Changing θ into –θ, the equation of the curve remains changed. Therefore the curve is symmetrical about the initial line. Again changing θ into π–θ, the equation remains unchanged, thus

Remarks

the curve is symmetrical about the line $\theta = \frac{\pi}{2}$ (the y-axis). Combining the two symmetries we see that the curve is symmetrical about the pole also. [It can be observed by changing θ into π + θ].

2. Origin : Putting r = 0, we get cos 2θ = 0

$$\therefore \qquad 2\theta = (2n+1)\frac{\pi}{2}$$

or $$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$ are the tangents at the origin.

3. Asymptote : The curve has no asymptote.

4. Region : Since $|\cos 2\theta| \le 1$

$\therefore$ Maximum value of r is a. Thus the whole curve lies within the circle of radius a.

r is maximum when $|\cos 2\theta| = 1$

i.e., $2\theta = 2n\pi$ or $\theta = n\pi$

i.e., $\theta = 0, \pi$

r is minimum, when $\cos 2\theta = -1$, then r is –ve

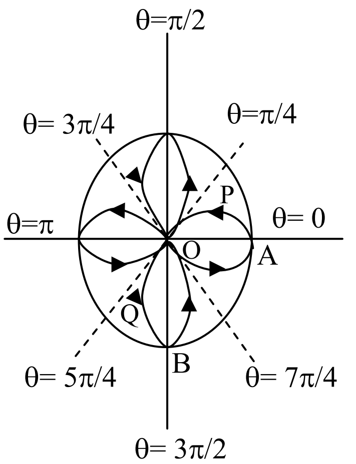
or $2\theta = \pi, 3\pi$

i.e., $$\theta = \frac{\pi}{2}, \frac{3\pi}{2}.$$

5. Table : Because of the symmetry about both the axes, we make table only for $0 \le \theta \le \frac{\pi}{2}$.

θ :	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r :	a	0.5a	0	–0.5a	–a

Thus the shape of the curve for this table is APOQB. Because of symmetry about both the axes, we complete the shape of the curve in other quadrants.



Note : The curve is traced as shown by the arrows starting from A and ending at A, without lifting the pen from the paper even once. This shows the continuity of the curve for $0 \le \theta \le 2\pi$.

Example 2 : Trace the curve r = a(1 + cosθ) [Cardioide].

Solution : The equation of the curve is r = a(1 + cosθ)

1. Symmetry : Changing θ into –θ, the equation of the curve remains unchanged. Thus the curve is symmetrical about the initial line (or x-axis).

2. Origin : When r = 0, we get 1 + cosθ = –1 i.e., θ = π is the tangent at the origin.

3. Asymptotes : The curve has no asymptotes as it is a closed curve.

4. Region : Since the maximum value of cosθ is 1, therefore the maximum value of r is 2a i.e. the whole of the curve lies within a circle of radius 2a.

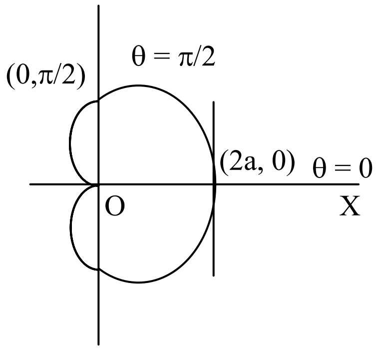
Remarks

5. Table : We find the corresponding values of θ for $0 \le \theta \le \pi$. The shape of the curve for the values $\pi \le \theta \le 2\pi$ shall be the same as for those of $0 \le \theta \le \pi$, because the curve is symmetrical about the initial line.

θ :	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r :	2a	1.86a	1.5a	a	0.5a	0.14a	0

6. Value of ϕ : We have $\frac{dr}{d\theta} = -a\sin\theta$

$\therefore$



$$\tan\phi = r\frac{d\theta}{dr} = -\frac{a(1+\cos\theta)}{a\sin\theta} = -\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$= -\cot\frac{\theta}{2} = \tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$$

$\therefore$ At $\theta = 0$, $\phi = \frac{\pi}{2}$ i.e., at (2a, 0) the tangent is perpendicular to the line $\theta = 0$.

Hence the shape of the curve is as shown in the above figure.

Example 3 : Trace the curve $r = a(1 - \sin\theta)$.

Solution : The given curve is $r = a(1 - \sin\theta)$.

1. Symmetric : Changing θ into $\pi-\theta$, the equation of the curve remains unchanged. Therefore the symmetry of the curve is about $\theta = \frac{\pi}{2}$.

2. Pole : Putting r = 0, we get $\sin\theta = 1$ or $\theta = \frac{\pi}{2}$. Thus $\theta = \frac{\pi}{2}$ is the tangent to the curve at the origin.

3. Asymptotes : The curve has no asymptote.

4. Region : r is a maximum when $\sin\theta = -1$ i.e., when $\theta = \frac{3\pi}{2}$ therefore the whole of the curve lies within a circle of radius 2a i.e., it is a closed curve.

Remarks

5. Value of ϕ : $\frac{dr}{d\theta} = -a\cos\theta$

$$\therefore \qquad \tan\phi = r\frac{d\theta}{dr} = -\frac{a(1-\sin\theta)}{a\cos\theta} = -\frac{\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)^2}{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}$$

$$= -\frac{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}} = -\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

$$= \cot\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$$

$$\therefore \qquad \tan\phi = \infty \text{ i.e., } \phi = \frac{\pi}{2} \text{ when } \cot\left(\frac{\pi}{4}+\frac{\theta}{2}\right) = \infty$$

$$\text{i.e.,} \qquad \frac{\pi}{4}+\frac{\theta}{2} = 0 \qquad \text{or} \qquad \pi$$

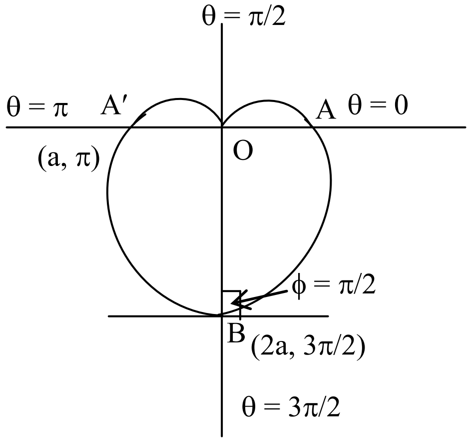
$$\text{i.e.,} \qquad \frac{\theta}{2} = \frac{3\pi}{4} \qquad \text{or} \qquad \theta = \frac{3\pi}{2}$$

Now when $\theta = \frac{3\pi}{2}$, then r = 2a.

6. Table :

θ :	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r :	2a	1.86a	1.5a	a	0.5a	0.14a	0

Since the curve is symmetrical about the line $\theta = \frac{\pi}{2}$ (y-axis), therefore the form of the curve from $\frac{3\pi}{2}$ to 2π and 0 to $\frac{\pi}{2}$ is the same as from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$. Hence the shape of the curve is as shown in the figure below :



Remarks

Example 4 : Trace the curve $r = a(1 + \sec\theta)$.

Solution : The equation of the curve is $r = \dfrac{a(1+\cos\theta)}{\cos\theta}$

1. Symmetry : When θ is changed into $-\theta$, the equation of the curve remains unchanged. Therefore the symmetry of the curve is about the initial line i.e., $\theta = 0$ (x-axis).

2. Pole or Origin : When $r = 0$, we get $\cos\theta = -1$ or $\theta = \pi$.

$\therefore \theta = \pi$ is the tangent to the curve at the origin.

3. Asymptote : The equation of the curve is of 4^{th} degree in x and y. Both y^4 and y^3 are absent. The co-efficient of y^2 is $(x - a)^2$. Therefore $x - a = 0$ or the line $x = a$ is an asymptote to the curve.

4. Region : Changing the equation into Cartesian co-ordinates i.e., putting $\cos\theta = \dfrac{x}{r}$, we get

$$r\cos\theta = a(1 + \cos\theta)$$

or $$x = a + a.\frac{x}{r}$$

or $$x - a = \frac{ax}{r}$$

or $$(x - a)^2 r^2 = a^2x^2$$

or $$(x - a)^2(x^2 + y^2) = a^2x^2$$

or $$(x - a)^2y^2 = a^2x^2 - x^2(x - a)^2 = x^3(2a - x)$$

or $$y^2 = \frac{x^3(2a-x)}{(x-a)^2}$$

Therefore, when $x < 0$, then y is imaginary

and when $x > 2a$, then y is again imaginary.

Thus the whole of the curve lies between the lines $x = 0$ and $x = 2a$. Also the origin is a cusp and $y = 0$ is cuspidal tangent.

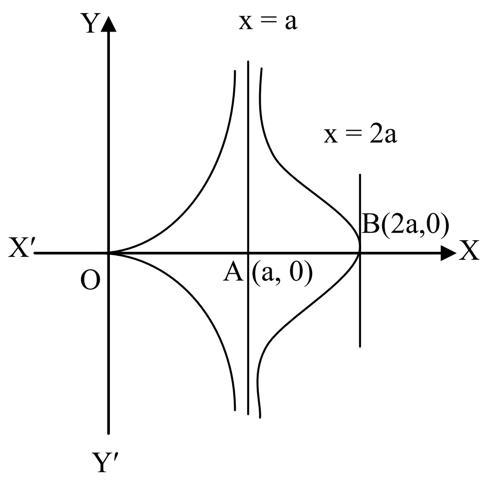
5. Intersection with the axes of co-ordinates : The curve meets x-axis when $y = 0$, which gives $x = 0$ and $x = 2a$. Shifting the origin at $(2a, 0)$, i.e., putting $x = X + 2a$, $y = Y + 0$, we get

$$(X + a)^2Y^2 = (X + 2a)^3(-X)$$

Remarks

The tangent at the new origin is X = 0. Therefore the line x = 2a is a tangent at (2a, 0).

Hence the shape of the curve is as shown in the figure below :



Exercise 5.3

Trace the following curves :

1. $r = a(1 + \sin\theta)$
2. $r = a(1 - \cos\theta)$
3. $r = a \sin 2\theta$
4. $r = a \cos 4\theta$

Keywords : Cartesian equation, parametric equation, polar equation.

Summary : The curve tracing is the shape of a curve without plotting the large number of points. Methods to tracing of Cartesian, parametric, polar curves. We check the following steps for tracing the curves.

1. Symmetry
2. Origin
3. Asymptotes
4. Points of intersection
5. Region
6. Special points.

CHAPTER – VI

REDUCTION FORMULAE

Remarks

6.0 STRUCTURE

6.1 INTRODUCTION

Using integration by parts or by method of connecting the integrals, it is possible to make an integral depend on another similar integral. The integration of the given integral can be completed by means of repeated application of the relation already established.

6.2 OBJECTIVE

After reading this lesion, you should be able to

- Understand the meaning of Reduction Formulae.
- Understand to connect an integral with another integral.

6.3 Definition. A formula which connects an integral with another integral of a simpler but of same type, is called a reduction formulae.

6.4 REDUCTION FORMULAE

6.4.1Find Reduction Formulae for $\int \sin^n xdx$

Solution. Let

$$
\begin{aligned}
I_n &= \int \sin^n xdx \\
&= \int \sin^{n-1} x.\sin xdx \\
&= \sin^{n-1}x(-\cos x) - \int (n-1)\sin^{n-2} x \cos x(-\cos x)dx \\
&= -\sin^{n-1}x \cos x + (n-1)\int \sin^{n-2} x \cos^2 xdx
\end{aligned}
$$

Remarks

$$= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x(1-\sin^2 x)dx$$

$$= -\sin^{n-1}x\cos x + (n-1)\int (\sin^{n-2}x - \sin^n x)dx$$

$$= -\sin^{n-1}x\cos x + (n-1)I_{n-2} - (n-1)I_n$$

$$I_n(1+n-1) = -\sin^{n-1}x\cos x + (n-1)I_{n-2}$$

$$I_n = -\frac{\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}I_{n-2}$$

6.4.2 Reduction formula for $\int \tan^n x\,dx$

$$\int \tan^n x\,dx = \int \tan^{n-2}x.\tan^2 x dx$$

$$= \int \tan^{n-2}x(\sec^2 x - 1)dx$$

$$= \int \tan^{n-2}x\sec^2 x dx - \int \tan^{n-2}x dx$$

$$= \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx$$

$$\left[\text{Using } \int [f(x)]^n.f'(x)dx = \frac{[f(x)]^{n+1}}{n+1}, n \neq 1\right]$$

Thus if $I_n = \int \tan^n x dx$, then $I_n = \dfrac{\tan^{n-1}x}{n-1} - I_{n-2}$

which is the required reduction formula.

6.4.3 Reduction formula for $\int \sec^n x\,dx$

$$\int \sec^n x\,dx = \int \sec^{n-2}x.\sec^2 x dx$$

$$= \sec^{n-2}x(\tan x) - \int (n-2)\sec^{n-3}x.\sec x\tan x(\tan x)dx$$

[Integrating by parts]

$$= \sec^{n-2}x\tan x - (n-2)\int \sec^{n-2}x.\tan^2 x dx$$

$$= \sec^{n-2}x\tan x - (n-2)\int \sec^{n-2}x(\sec^2 x - 1)dx$$

$$= \sec^{n-2}x\tan x - (n-2)\int \sec^n x dx + (n-2)\int \sec^{n-2}x dx$$

$$\therefore \quad \int \sec^n x\,dx + (n-2)\int \sec^n x\,dx = \sec^{n-2}x.\tan x + (n-2)\int \sec^{n-2}x\,dx$$

$$\therefore \quad (n-1)\int \sec^n x\,dx = \sec^{n-2}x.\tan x + (n-2)\int \sec^{n-2}x\,dx$$

$$\therefore \quad \int \sec^n x\,dx = \frac{\sec^{n-2}x\tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx$$

Thus if $I_n = \int \sec^n x\,dx$, then $I_n = \dfrac{\sec^{n-2}x\tan x}{n-1} + \dfrac{n-2}{n-1}I_{n-2}$ …(1)

which is the required reduction formula.

6.4.4 Reduction formula for $\int x^n e^{ax}dx$

$$\int x^n e^{ax}dx = x^n\frac{e^{ax}}{a} - \int nx^{n-1}\frac{e^{ax}}{a}dx$$

Remarks

$$= \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1} e^{ax} dx$$

Thus if $I_n = \int x^n e^{ax}$, then $I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$

which is the required reduction formula.

6.4.5 Reduction formula for $\int x^m (\log x)^n dx$

$$\int x^m (\log x)^n dx = \frac{x^{m+1}}{m+1}.(\log x)^n - \int \frac{x^{m+1}}{m+1}.\frac{n(\log x)^{n-1}}{x}$$

[Integrating by parts]

$$= \frac{x^{m+1}}{m+1}(\log x)^n - \frac{n}{m+1}\int x^m (\log x)^{n-1} dx$$

…(1)

which is the required reduction formula.

To evaluate $\int_0^1 x^4 (\log x)^3 dx$ using reduction formula for $\int x^n (\log x)^n dx$

Putting m = 4 in (1), we get

$$\int x^4 (\log x)^n dx = \frac{x^5}{5}(\log x)^n - \frac{n}{5}\int x^4 (\log x)^{n-1} dx$$

Putting n = 3, 2, 1 in (2) respectively, we get

$$\int x^4 (\log x)^3 dx = \frac{x^5}{5}(\log x)^3 - \frac{3}{5}\int x^4 (\log x)^2 dx$$

$$\int x^4 (\log x)^2 dx = \frac{x^5}{5}(\log x)^3 - \frac{2}{5}\int x^4 (\log x) dx$$

$$\int x^4 (\log x) dx = \frac{x^5}{5}(\log x) - \frac{1}{5}\int x^4 dx$$

$$= \frac{x^5}{5}(\log x) - \frac{x^5}{25}.$$

From (3), (4) and (5), we get

$$\int x^4 (\log x)^3 dx = \frac{x^5}{5}(\log x)^3 - \frac{3x^5}{25}(\log x)^2 + \frac{6}{125}x^5 \log x - \frac{6}{625}x^5$$

$$\therefore \qquad \int_0^1 x^4 (\log x)^3 dx = -\frac{6}{625}$$

[$\because$ As $x \to 0$, $x^5(\log x)^3$, $x^5(\log x)^2$, $x^5 \log x$ all tended to zero]

Example 1. Evaluate $\int_0^\infty x^n e^{-x} dx$, where n is a positive integer.

Solution. Let $I_n = \int_0^\infty x^n e^{-x} dx$ …(1)

Integrating by parts, we get

Remarks

$$I_n = \left|\frac{x^n e^{-x}}{-1}\right| - \int_0^\infty n x^{n-1} \frac{e^{-x}}{-1} dx$$

$$= n\int_0^\infty x^{n-1} e^{-x} dx \qquad [\because x^n e^{-x} \to \text{ and } x \to \infty]$$

$$I_n = n\, I_{n-1} \qquad \ldots(2)$$

Replacing n by (n–1) in (2), we get

$$I_{n-1} = (n-1)\, I_{n-2} \qquad \ldots(3)$$

Using (3) in (2), we have

$$I_n = n(n-1)\, I_{n-2}$$

In general, $I_n = [n(n-1)(n-2)\ldots\ldots n \text{ factors}]\, I_{n-n}$

$$= n!\, I_0 \qquad \ldots(4)$$

From (1), $I_0 = \int_0^\infty x^0 e^{-x} dx = \int_0^\infty e^{-x} dx$

$$I_0 = \left|\frac{e^{-x}}{-1}\right|_0^\infty = -[e^{-\infty} - e^0] = -(0-1) = 1$$

$\therefore$ From (4), we get $I_n = n!$.

Example 2. If $u_n = \int_0^{\pi/4} \tan^n x dx$, shows that $u_n + u_{n-2} = \frac{1}{n-1}$. Hence evaluate u_5.

$$u_n = \int_0^{\pi/4} \tan^n x dx = \int_0^{\pi/4} \tan^{n-2} x \tan^2 x\, dx$$

$$= \int_0^{\pi/4} \tan^{n-2}(\sec^2 x - 1)dx$$

$$= \int_0^{\pi/4} \tan^{n-2} x \sec^2 x\, dx - \int_0^{\pi/4} \tan^{n-2} x\, dx$$

$$= \left|\frac{\tan^{n-1} x}{n-1}\right|_0^{\pi/4} - \int_0^{\pi/4} \tan^{n-2} x\, dx$$

$$= \frac{1}{n-1}\left[\tan^{n-1}\frac{\pi}{4} - \tan^{n-1} 0\right] - u_{n-2}$$

i.e., $$u_n = \frac{1}{n-1} - u_{n-2}$$

$$\left[\because \tan^{n-1} 0 = 0; \tan^{n-1}\frac{\pi}{4} = 1\right]$$

$\therefore$ $$u_n + u_{n-2} = \frac{1}{n-1} \qquad \ldots(1)$$

For calculating u_5, put n = 5 in (1)

$\therefore$ $$u_5 + u_3 = \frac{1}{4} \qquad \ldots(2)$$

Remarks

Putting n = 3 in (1), we get

$$u_3 + u_1 = \frac{1}{2}$$

$$\therefore \quad u_3 = \frac{1}{2} - \int_0^{\pi/4} \tan x\, dx$$

$$= \frac{1}{2} - [\log \sec x]_0^{\pi/4} = \frac{1}{2} - \log\sqrt{2} = \left(\frac{1}{2} - \frac{1}{2}\log 2\right)$$

$\therefore$ From (2), we get

$$u_5 = \frac{1}{4} - u_3 = \frac{1}{4} - \left(\frac{1}{2} - \frac{1}{2}\log 2\right)$$

$$= \frac{1}{2}\log 2 - \frac{1}{4}\ .$$

Example 3. If $I2_n = \int_0^1 x^n (\log x)^n\, dx$ ($n \geq 0$ and m is positive integer); prove that $I_m = \frac{m}{n+1} I_{m-1}$ and hence evaluate I_3.

Solution.

$$I_m = \int_0^1 x^n (\log x)^m\, dx$$

$$= \left|\frac{x^{n+1}}{n+1}(\log x)^m\right|_0^1 - \int_0^1 \frac{x^{n+1}}{n+1}.m\frac{(\log x)^{m-1}}{x}\, dx \text{ [Integrating by parts]}$$

$$= (0-0) - \frac{m}{n+1}\int_0^1 x^n.(\log x)^{m-1} dx$$

$$\therefore \quad I_m = -\frac{m}{n+1} I_{m-1} \qquad \ldots(1)$$

Putting m = 3,2,1 successively in (1), we get

$$I_3 = -\frac{3}{n+1} I_2 \qquad \ldots(2)$$

$$I_2 = -\frac{2}{n+1} I_1 \qquad \ldots(3)$$

$$I_1 = -\frac{1}{n+1} I_0 = -\frac{1}{n+1}\int_0^1 x^n dx$$

$$= -\frac{1}{n+1}\left|\frac{x^{n+1}}{n+1}\right|_0^1$$

$$= -\frac{1}{n+1}\left[\frac{1}{n+1} - 0\right] = -\frac{1}{(n+1)^2}$$

$$\therefore \quad I_1 = -\frac{1}{(n+1)^2} \qquad \ldots(4)$$

From (2), (3) and (4), we get

Remarks

$$I_3 = \left(-\frac{3}{n+1}\right)\left(-\frac{2}{n+1}\right)\left(-\frac{1}{(n+1)^2}\right) = -\frac{6}{(n+1)^4}\ .$$

Exercise 6.1

1. If $I_n = \int_0^{\pi/4} \tan^n \theta$, prove that $I_n + I_{n-2} = \frac{1}{n-1}$. Hence prove that

$$\int_0^a x^5 (2a^2 - x^2)^{-3} dx = \frac{1}{2}\left(\log 2 - \frac{1}{2}\right).$$

2. Find the reduction formula of the following

(i) $\int \cos^n x\, dx$ (ii) $\int \cot^n x\, dx$ (iii) $\int \operatorname{cosec}^n x\, dx$

3. Obtain a reduction formula for $\int x^m (\log x)^n dx$ and hence show that

$$\int_0^1 x^m (\log x)^n dx - \frac{(-1)^n n!}{(m+1)^{n+1}}$$, where $m \geq 0$ and n is a positive integer.

6.5.1 Reduction formula for $\int e^{ax} \sin^n bx\, dx$

$$\int e^{ax} \sin^n dx\, dx = \sin^n dx\left(\frac{e^{ax}}{a}\right) - \int n \sin^{n-1} bx \cos bx.b.\left(\frac{e^{ax}}{a}\right) dx$$

[Integrating by parts]

$$= \frac{e^{ax}}{a}\sin^n bx - \frac{nb}{a}\int \sin^{n-1} bx \cos bx.e^{ax} dx$$

$$= \frac{e^{ax}}{a}\sin^n bx - \frac{nb}{a}\left(\sin^{n-1}.bx \cos bx.\frac{e^{ax}}{a}\right.$$

$$\left.-\int[(n-1)\sin^{n-2} bx.\cos bx.b\cos bx + \sin^{n-1} bx(-\sin bx.b)\frac{e^{ax}}{a} dx\right)$$

$$= \frac{e^{ax}}{a}\sin^n bx - \frac{nb}{a}\left[\frac{e^{ax}\sin^{n-1} bx \cos bx}{a}\right.$$

$$\left.-\int[(n-1)b\sin^{n-2} bx \cos^2 bx - b\sin^n bx]\frac{e^{ax}}{a} dx\right]$$

$$= \frac{e^{ax}}{a}\sin^n bx - \frac{nb}{a^2} e^{ax} \sin^{n-1} bx \cos bx$$

$$+ \frac{nb^2}{a^2}(n-1)\int e^{ax} \sin^{n-2} bx(1-\sin^2 bx) dx - \frac{nb^2}{a^2}\int e^{ax}.\sin^n bx\, dx$$

$$= \frac{ae^{ax}\sin^n bx - nbe^{ax}\sin^{n-1} bx \cos bx}{a^2} + \frac{nb^2}{a^2}(n-1)\int e^{ax} \sin^{n-2} bxdx$$

$$-n(n-1)\frac{b^2}{a^2}\int e^{ax} \sin^n bx\, dx - \frac{nb^2}{a^2}\int e^{ax} \sin^n bx\, dx$$

Remarks

$$\therefore \quad \int e^{ax}\sin^n bx\,dx = \frac{ae^{ax}\sin^n bx - nbe^{ax}\sin^{n-1} bx\cos bx}{a^2} + \frac{n(n-1)b^2}{a^2}\int e^{ax}\sin^{n-2} bx\,dx - \frac{n^2b^2}{a^2}\int e^{ax}\sin^n bx\,dx$$

$$\therefore \left(1+\frac{n^2b^2}{a^2}\right)\int e^{ax}\sin^n bx\,dx = \frac{ae^{ax}\sin^n bx - nbe^{ax}\sin^{n-1} bx\cos bx}{a^2} + \frac{n(n-1)b^2}{a^2}\int e^{ax}\sin^{n-2} bx\,dx$$

$$\therefore \quad \int e^{ax}\sin bx\,dx = \frac{ae^{ax}\sin^n bx - nbe^{ax}\sin^{n-1} bx\cos bx}{a^2+n^2b^2} + \frac{n(n-1)b^2}{a^2+n^2b^2}\int e^{ax}\sin^{n-2} bx\,dx$$

6.5.2 Reduction formula for $\int \cos^m x \sin nx\,dx$

$$\int \cos^m x\sin nx\,dx = \cos^m x\left(-\frac{\cos nx}{n}\right) - \int m\cos^{m-1} x(-\sin x).\left(-\frac{\cos nx}{n}\right)dx$$

[Integrating by parts]

$$= -\frac{\cos^m x\cos nx}{n} - \int \cos^{m-1} x\sin x.\cos nx\,dx \qquad \ldots(1)$$

Now $\quad \sin(nx - x) = \sin nx \cos x - \cos nx \sin x$

$\therefore \quad \cos nx \sin x = \sin nx \cos x - \sin(n-1)x$

$\therefore$ From (1), we get

$$\int \cos^m x\sin nx\,dx = -\frac{\cos^m x\cos nx}{n} - \frac{m}{n}\int \cos^{m-1} x[\sin nx\cos x - \sin(n-1)x]dx$$

$$= -\frac{\cos^m x\cos nx}{n} - \frac{m}{n}\int \cos^m x\sin nx\,dx + \frac{m}{n}\int \cos^{m-1} x\sin(n-1)x dx$$

$$\therefore \quad \left(1+\frac{m}{n}\right)\int \cos^m x\sin nx\,dx = -\frac{\cos^m x\cos nx}{n} + \frac{m}{n}\int \cos^{m-1} x\sin(n-1)x\,dx$$

$$\therefore \quad \int \cos^m x\sin nx\,dx = -\frac{\cos^m x\cos nx}{m+n} + \frac{m}{m+n}\int \cos^{m-1} x\sin(n-1)x\,dx$$

or $\quad I_{m,n} = -\dfrac{\cos^m x\cos nx}{m+n} + \dfrac{m}{m+n} I_{m-1,n-1}$

6.5.3 Reduction formula for $\int \frac{\sin nx}{\cos x}dx$

Consider $\quad \dfrac{\sin nx}{\cos x} + \dfrac{\sin(n-2)x}{\cos x} = \dfrac{\sin nx + \sin(n-2)x}{\cos x}$

$\therefore \quad \dfrac{\sin nx}{\cos x} + \dfrac{\sin(n-2)x}{\cos x} = \dfrac{2\sin(n-1)x\cos x}{\cos x}$

or $\quad \dfrac{\sin nx}{\cos x} + \dfrac{\sin(n-2)x}{\cos x} = 2\sin(n-1)x$

Integrating both sides w.r.t x, we get

$$\int \frac{\sin nx}{\cos}dx + \int \frac{\sin(n-2)x}{\cos x}dx = 2\int \sin(n-1)x\,dx = -2.\frac{\cos(n-1)x}{(n-1)}$$

Remarks

$\therefore \quad \int \frac{\sin nx}{\cos x} dx = -\frac{2\cos(n-1)x}{(n-1)} - \int \frac{\sin(n-2)x}{\cos x} dx$

which is the required reduction formula.

Example 1. (i) If $I_{m,n} = \int \cos^m x \cos nx$, prove that

$$(m+n)\, I_{m,n} = \cos^m x \sin nx + m\, I_{m-1,n-1}$$

(ii) If $I_{m,n} = \int_0^{\pi/2} \cos^m x \cos nx\, dx$, prove that

$$I_{m,n} = \frac{m}{m+n} I_{m-1,n-1}$$

Solution. (i) $\int \cos^m x \cos nx = \cos^m x.\frac{\sin nx}{n} - \int m\cos^{m-1} x.(-\sin x)\frac{\sin nx}{n}.dx$

[Integrating by parts]

$$= \frac{\cos^m x \sin nx}{n} + \frac{m}{n}\int \cos^{m-1} x(\sin nx \sin x)dx \qquad \ldots(1)$$

Now $\cos(n-1)x = \cos nx.\cos x + \sin nx.\sin x$

$\therefore \quad \sin nx \sin x = \cos(n-1)x - \cos nx \cos x$

From (1), we get

$$\int \cos^m x \cos nx\, dx = \frac{\cos^m \sin nx}{n} + \frac{m}{n}\int \cos^{m-1} x \cos(n-1)x dx - \frac{m}{n}\int \cos^m x\, nx\, dx$$

$$\therefore \quad \left(1+\frac{m}{n}\right)\int \cos^m x \cos nx\, dx = \frac{\cos^m x \sin nx}{n} + \frac{m}{n}\int \cos^{m-1} x.\cos(n-1)x\, dx$$

or $\quad (m+n)\int \cos^m x \cos nx\, dx = \cos^m x \sin nx + m\int \cos^{m-1} x \cos(n-1)x\, dx \qquad \ldots(2)$

or $\quad (m+n)I_{m,n} = \cos^m x \sin nx + m\, I_{m-1,n-1}$

(ii) Using (2), we have

$$(m+n)\int_0^{\pi/2} \cos^m x \cos nx\, dx = |\cos^m x \sin nx|_0^{\pi/2} + m\int_0^{\pi/2} \cos^{m-1} x.\cos(n-1)x\, dx$$

$\Rightarrow \quad (m+n)\, I_{m,n} = m\, I_{m-1,n-1}$

or $\quad I_{m,n} = \frac{m}{m+n} I_{m-1,n-1}$.

Example 2. If $u_n = \int \cos n\theta \operatorname{cosec}\theta\, d\theta$, prove that $u_n - u_{n-2} = \frac{2\cos(n-1)\theta}{n-1}$.

Solution. Consider,

$$\frac{\cos n\theta}{\sin\theta} - \frac{\cos(n-2)\theta}{\sin\theta} = \frac{\cos n\theta - \cos(n-2)\theta}{\sin\theta}$$

$$= \frac{2\sin(n-1)\theta(\sin(-\theta)}{\sin\theta}$$

$$\frac{\cos n\theta}{\sin\theta} - \frac{\cos(n-2)\theta}{\sin\theta} = -2\sin(n-1)\theta$$

Remarks

$$\Rightarrow \qquad \int \frac{\sin n\theta}{\sin\theta}d\theta - \int \frac{\cos(n-2)\theta}{\sin\theta}d\theta = 2\int -\sin(n-1)\theta\, d\theta$$

$$\int \frac{\cos n\theta}{\sin\theta}d\theta - \int \frac{\cos(n-2)\theta}{\sin\theta}d\theta = 2\frac{\cos(n-1)\theta}{n-1}$$

$$u_n - u_{n-2} = \frac{2\cos(n-1)\theta}{n-1}.$$

Example 5. Shows that $\int_0^{\pi} \frac{\sin nx}{\sin x}dx = 0$ or π according as n is even or odd positive integer.

Solution.

$$\frac{\sin nx}{\sin x} - \frac{\sin(n-2)x}{\sin x} = \frac{2\cos(n-1)x\sin x}{\sin x}$$

$$\therefore \qquad \frac{\sin nx}{\sin x} - \frac{\sin(n-2)x}{\sin x} = 2\cos(n-1)x$$

Taking the integral of both sides, we have

$$\int \frac{\sin nx}{\sin x}dx - \int \frac{\sin(n-2)x}{\sin x}dx = 2\int \cos(n-1)x\, dx$$

or

$$\int \frac{\sin nx}{\sin x}dx = 2\frac{\sin(n-1)x}{n-1} + \int \frac{\sin(n-2)x}{\sin x}dx$$

Inserting the limits from 0 to π, we get

$$\int_0^{\pi} \frac{\sin nx}{\sin x}dx = 0 + \int_0^{\pi} \frac{\sin(n-2)x}{\sin x}dx$$

$$I_n = I_{n-2}$$

Changing n to n–2, we get

$$I_{n-2} = I_{n-4}$$

$$\therefore \qquad I_n = I_{n-2} = I_{n-4}$$

Case I. When n is even :

$$I_n = I_{n-2} = I_{n-4} = \ldots = I_0$$

But

$$I_0 = \int_0^{\pi} \frac{\sin 0}{\sin x}dx = 0$$

$$\therefore \qquad \int_0^{\pi} \frac{\sin nx}{\sin x}dx = 0, \text{ when n is even.}$$

Case II. When n is odd :

$$I_n = I_{n-2} = I_{n-4} = \ldots = I_1$$

But

$$I_1 = \int_0^{\pi} \frac{\sin nx}{\sin x}dx = \pi, \text{ when n is odd.}$$

Exercise 6.2

1. If $I_n = \int_0^{\pi/2} x^n \sin x\, dx$ and $n > 1$. Then show that $I_n + n(n-1)I_{n-2} = n(\pi/2)^{n-1}$. Hence evaluate I_6.

Remarks

2. If $I_{m,n} = \int_0^{\pi/2} \cos^m x \sin nx\, dx$, prove that $I_{m,n} = \frac{1}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$. Hence evaluate $I_{5,3}$.

3. Show that $\int (\sin^{-1} x)^n dx = x(\sin^{-1} x)^n + n\sqrt{1-x^2}(\sin^{-1} x)^{n-1} - n(n-1)\int (\sin^{-1} x)^{n-2} dx$

4. Obtain a reduction formula for $\int x^n \cos x\, dx$ and hence evaluate $\int x^3 \cos x\, dx$.

Ans. $\int x^n \cos x\, dx = x^n \sin x + nx^{n-1}\cos x - n(n-1)\int x^{n-2}\cos x\, dx$; $(x^3 - 6x) \sin x + (3x^2 - 6) \cos x$.

5. If $I_n = \int_0^{\pi/2} x^n \sin mx\, dx$, prove that $I_n = \frac{n\pi^{n-1}}{m^2 . 2^{n-1}} - \frac{n(n-1)}{m^2} I_{n-2}$, where m is an integer of the form 4r + 1.

Evaluation of $\int_0^{\pi/2} \sin^n x dx$, n = positive even and odd integer

Solution. We know that,

$$\int \sin^n x dx = \frac{-\sin^{n-1}\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x dx$$

$$\Rightarrow \int_0^{\pi/2} \sin^n x dx = 0 + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x dx$$

$$\Rightarrow \int_0^{\pi/2} \sin^n x dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x dx$$

$$I_n = \frac{n-1}{n} I_{n-2}$$

On changing n to (n–2), we get

$$I_{n-2} = \frac{(n-2)-1}{n-2} I_{n-4}$$

$$\therefore \quad I_n = \frac{n-1}{n} . \frac{n-3}{n-2} I_{n-4}$$

And so on.

Case I. When n is a +ve even integer :

$$I_n = \frac{n-1}{n} . \frac{n-3}{n-2} . \frac{n-5}{n-4} ... \frac{1}{2} I_0$$

But $$I_0 = \int_0^{\pi/2} \sin^0 x\, dx = \int_0^{\pi/2} 1\, dx = \frac{\pi}{2}$$

$$\therefore \quad I_n = \frac{n-1}{n} . \frac{n-3}{n-2} . \frac{n-5}{n-4} ... \frac{1}{2} . \frac{\pi}{2} \quad ...(1)$$

Case II. When n is a +ve odd integer :

$$I_n = \frac{n-1}{n} . \frac{n-3}{n-2} . \frac{n-5}{n-4} ... \frac{2}{3} I_1$$

Remarks

But $\quad I_1 = \int_0^{\pi/2} \sin x\, dx = |-\cos x|_0^{\pi/2} = 1$

$$\therefore \quad I_n = \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}...\frac{2}{3} \qquad ...(2)$$

Remark. The above formula is called Walli's formula and can be expressed as

$$\int_0^{\pi/2} \sin^n x dx = \frac{(n-1)\times \text{go on diminishing by } 2}{n \times \text{go on diminishing by } 2}.\left(\frac{\pi}{2}\right)^*$$

Reduction Formula for $\int \cos^n x\, dx$

Similarly for $\int \cos^n x\, dx$.

Example 1. Prove that $\int_0^{\pi/2} \sin^{2n} x\, dx = \frac{2n!}{[2^n (n!)]^2}.\frac{\pi}{2}$.

Solution. We know that 2n is even integer

$$\int_0^{\pi/2} \sin^{2n} x\, dx = \frac{(2n-1)(2n-3)...1}{2n(2n-2)...2} \times \frac{\pi}{2}$$

Multiplying both numerator and denominator by denominator ie., by 2n(2n–2)…2, we have

$$\int_0^{\pi/2} \sin^{2n} x\, dx = \frac{2n(2n-1)(2n-2)(2n-3)...2.1}{[2n(2n-2)...2]^2}.\frac{\pi}{2}$$

$$= \frac{2n!}{[2.n.2(n-1)...2.1]^2}.\frac{\pi}{2}$$

$$= \frac{2n!}{[2^n n(n-1)...1]^2}.\frac{\pi}{2} = \frac{2n!}{[2^n (n!)]^2}.\frac{\pi}{2} .$$

Example 2. Evaluate :

(i) $\int_0^{2\pi} \sin^7 \frac{\theta}{4} d\theta$ $\qquad$ (ii) $\int_0^{\pi/2} \cos^5 \theta\, d\theta$

(i) $\quad I = \int_0^{2\pi} \sin^7 \frac{\theta}{4} d\theta$

Put $\frac{\theta}{4} = z \quad \Rightarrow \quad \theta = 4z$ so that $d\theta = 4dz$

$$\therefore \quad I = 4\int_0^{\pi/2} \sin^7 z dz = 4 \times \frac{6.4.2}{7.5.3} = \frac{64}{35}$$

(ii) $\quad \int_0^{\pi/2} \cos^5 \theta\, d\theta = \frac{4.2}{5.3} = \frac{8}{15}$.

Remarks

Example 5. If $u_n = \int_0^{\pi/2} \theta \sin^n \theta \, d\theta$ (n>1); prove that $u_n = \frac{n-1}{n} u_{n-2} + \frac{1}{n^2}$ and deduce that $u_5 = \frac{149}{225}$.

Solution. $u_n = \int_0^{\pi/2} \theta \sin^n \theta d\theta = \int_0^{\pi/2} \theta \sin^{n-1} \theta . \sin \theta d\theta$

Integrating by parts, we have

$$u_n = |(\theta \sin^{n-1} \theta)(-\cos\theta)|_0^{\pi/2}$$

$$-\int [1.\sin^{n-1}\theta + \theta.(n-1)\sin^{n-2}\theta\cos\theta] \times (-\cos\theta) d\theta$$

$$= 0 + \int_0^{\pi/2} \sin^{n-1}\theta\cos\theta d\theta + (n-1)\int_0^{\pi/2} \theta\sin^{n-2}\theta\cos^2\theta d\theta$$

$$= \left|\frac{\sin^n\theta}{n}\right|_0^{\pi/2} + (n-1)\int_0^{\pi/2} \theta\sin^{n-2}\theta(1-\sin^2\theta)d\theta$$

$$= \frac{1}{n} + (n-1)\int_0^{\pi/2} \theta\sin^{n-2}\theta d\theta - (n-1)\int_0^{\pi/2} \theta\sin^n\theta d\theta$$

$\therefore \quad u_n + (n-1)u_n = \frac{1}{n} + (n-1)u_{n-2}$

$\therefore \quad u_n = \frac{1}{n^2} + \frac{n-1}{n} u_{n-2} \quad \ldots(1)$

Putting n = 5 in (1), we get

$$u_5 = \frac{1}{25} + \frac{4}{5} u_3 \quad \ldots(2)$$

Putting n = 3 in (2), we get

$$u_3 = \frac{1}{9} + \frac{2}{3} u_1$$

$$= \frac{1}{9} + \frac{2}{3}\int_0^{\pi/2} \theta\sin\theta d\theta \quad \ldots(3)$$

$\therefore$ From (2) and (3), we get

$$u_5 = \frac{1}{25} + \frac{4}{5}\left[\frac{1}{9} + \frac{2}{3}\int_0^{\pi/2} \theta\sin\theta \, d\theta\right]$$

$$= \frac{1}{25} + \frac{4}{45} + \frac{8}{15}\left[|\theta(-\cos\theta)|_0^{\pi/2} - \int_0^{\pi/2} 1.(-\cos\theta)d\theta\right]$$

$$= \frac{1}{25} + \frac{4}{45} + \frac{8}{15}[0 + |\sin\theta|_0^{\pi/2}]$$

$$= \frac{1}{25} + \frac{4}{45} + \frac{8}{15} = \frac{9+20+120}{225} = \frac{149}{225}.$$

Remarks

Exercise 6.3

1. Evaluate (i) $\int_0^1 \frac{x^3}{\sqrt{1-x^2}}dx$ (ii) $\int_0^{2a} \frac{x^{7/2}}{\sqrt{2a-x}}dx$

Ans. (i) $\frac{2}{3}$ (ii) $\frac{35\pi a^4}{8}$

2. Evaluate $\int_0^{\infty} \frac{dx}{(1+x^2)^n}$

Ans. $\frac{(2n-3)(2n-5)...3.1}{(2n-2)(2n-4)...4.2} \times \frac{\pi}{2}$

3. Evaluate $\int_0^1 \frac{x^{2n}}{\sqrt{1-x^2}}dx$ and hence find the sum of the series

$$\frac{1}{2n+1} + \frac{1}{2} \cdot \frac{1}{2n+3} + \frac{1.3}{2.4} \cdot \frac{1}{2n+5} + ...\infty .$$

Ans. $\frac{(2n-1)(2n-3)...1}{2n(2n-2)...2} \cdot \frac{\pi}{2}$

4. If $I_n = \int_0^{\pi/2} x \cos^n x dx$, prove that $I_n = \frac{n-1}{n} I_{n-2} - \frac{1}{n^2}$.

5. Evaluate :

(i) $\int_0^{\pi} \sin^4 x . \frac{\sqrt{1-\cos x}}{(1+\cos x)^2} dx$ (ii) $\int_0^a \frac{x^n}{\sqrt{ax-x^2}}dx$

(iii) $\int_0^{3\pi/2} \cos^5 \frac{\theta}{3} d\theta$ (iv) $\int_0^{\pi/4} (\cos 2\theta)^{3/2} \cos\theta d\theta$

Ans. (i) $\frac{64\sqrt{2}}{15}$ (ii) $\frac{1.3.5...(2n-1)}{2.4.6...2n} . \pi a^n$ (iii) $\frac{8}{5}$ (iv) $\frac{3\pi}{16\sqrt{2}}$.

6.7.1 Reduction formula for $\int \sin^m x \cos^n x\, dx$

$\int \sin^m x \cos^n x\, dx$ can be connected with any one of the following six integrals depending upon the sign of m and n in the integral to be connected

(i) $\int \sin^{m-2} x \cos^n x\, dx$ (if m is +ve)

(ii) $\int \sin^m x \cos^{n-2} x\, dx$ (if n is +ve)

(iii) $\int \sin^{m+2} x \cos^n x\, dx$ (if m is –ve)

(iv) $\int \sin^m x \cos^{n+2} x\, dx$ (if n is –ve)

(v) $\int \sin^{m-2} x \cos^{n+2} x\, dx$ (if m is +ve and n is –ve)

(vi) $\int \sin^{m+2} x \cos^{n-2} x\, dx$ (if n is +ve and m is –ve)

Note that we cannot connect the given integral to that in which the indices of both sin x and cos x are increased or decreased by 2.

Remarks

6.7.2 Method to connect $\int \sin^m x \cos^n x\, dx$ with any integral of the above type

(1) Let P = $\sin^{\alpha+1} x \cos^{\beta+1} x$, where α is smaller of the indices of sin x and β is smaller of the indices of cos x in two integrals to be connected.

(2) Differentiate P w.r.t. x to get $\frac{dP}{dx}$. Express it in terms of integrands of the two integrals to be connected.

(3) Integrate both sides w.r.t x and solve to get the value of the given integral.

Example 1. Connect $\int \sin^m x \cos^n x\, dx$ with $\int \sin^m x \cos^{n-2} x\, dx$.

Solution. Let $P = \sin^{m+1} x \cos^{n-2+1} x$

$$= \sin^{m+1} x \cos^{n-1} x$$

$$\frac{dP}{dx} = (m+1)\sin^m x.\cos x.\cos^{n-1}x + \sin^{m+1} x(n-1) \cos^{n-2}x(-\sin x)$$

$$= (m + 1) \sin^m x \cos^n x - (n - 1)\sin^{m+2} x \cos^{n-2} x$$

$$= (m + 1)\sin^m x \cos^n x - (n - 1) \sin^m x.\sin^2 x \cos^{n-2} x$$

$$= (m + 1)\sin^m x \cos^n x - (n - 1) \sin^m x (1 - \cos^2 x) \cos^{n-2} x$$

$$= (m+1) \sin^m x \cos^n x - (n-1) \sin^m x \cos^{n-2}x + (n-1)\sin^m x \cos^n x$$

$$= (m + n) \sin^m x \cos^n x - (n - 1) \sin^m x \cos^{n-2} x$$

Integrating both sides, we get

$$P = (m + n) \int \sin^m x \cos^n x\, dx - (n-1)\int \sin^m x \cos^{n-2} x\, dx$$

or

$$(m + n) \int \sin^m x \cos^n x\, dx = P + (n-1)\int \sin^m x \cos^{n-2} x\, dx$$

$$\therefore \quad \int \sin^m x \cos^n x\, dx = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n}\int \sin^m x \cos^{n-2} dx$$

which is the required reduction formula.

Example 2. Evaluate $\int_0^{\pi/2} \sin^m x \cos^n x\, dx$, where m and n are positive integers and hence find $\int_0^{\pi/2} \sin^5 x \cos^4 x\, dx$.

Solution. $I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x\, dx$ is connected with

$$I_{m-2,n} = \int_0^{\pi/2} \sin^{m-2} x \cos^n dx$$

Let $P = \sin^{m-1} x \cos^{n+1} x$

Differentiating w.r.t. x, we get

$$\frac{dP}{dx} = (m-1)\sin^{m-2}x.\cos x.\cos^{n+1}x + \sin^{m+1}x.(n+1)\cos^n x(-\sin x)$$

$$= (m-1)\sin^{m-2}x.\cos^{n+2}x - (n+1) \sin^m x.\cos^n x$$

$$= (m-1)\sin^{m-2}x.\cos^n x(1 - \sin^2 x) - (n+1) \sin^m x.\cos^n x$$

Remarks

$= (m-1)\sin^{m-2}x.\cos^n x - (m-1)\sin^m x\cos^n x - (n+1)\sin^m x.\cos^n x$

$= (m-1)\sin^{m-2} x\cos^n x - (m+n)\sin^m x.\cos^n x$

Integrating both sides, we get

$$P = (m-1)\int \sin^{m-2} x\cos^n x\,dx - (m+n)\int \sin^m x.\cos^n x\,dx$$

$$\therefore \qquad \int \sin^m x\cos^n dx = -\frac{\sin^{m-1} x.\cos^{n+1} x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2} x\cos^n x\,dx$$

Inserting the limits between 0 to $\frac{\pi}{2}$, we have

$$\int_0^{\pi/2} \sin^m x\cos^n x\,dx = \left|-\frac{\sin^{m-1} x\cos^{n+1} x}{m+n}\right|_0^{\pi/2} + \frac{m-1}{m+n}\int_0^{\pi/2} \sin^{m-2} x\cos^n x\,dx$$

$$\therefore \qquad I_{m,n} = \frac{m-1}{m+n} I_{m-2,n} \qquad \ldots(1)$$

Changing m to m – 2,

$$I_{m-2,n} = \frac{m-3}{m+n-2} I_{m-4,n} \qquad \ldots(2)$$

From (1) and (2), we get

$$I_{m,n} = \frac{m-1}{m+n}.\frac{m-3}{m+n-2}.I_{m-4,n}$$

Case I. When m is a +ve odd integer :

$$I_{m,n} = \frac{m-1}{m+n}.\frac{m-3}{m+n-2}\ldots\frac{2}{n+3} I_{1,n}$$

But $\quad I_{1,n} = \int_0^{\pi/2} \sin x\cos^n x\,dx = -\left|\frac{\cos^{n+1} x}{n+1}\right|_0^{\pi/2}$

$$= -\left[0 - \frac{1}{n+1}\right] = \frac{1}{n+1}$$

$$\therefore \qquad I_{m,n} = \frac{m-1}{m+n}.\frac{m-3}{m+n-2}\ldots\frac{2}{n+3}.\frac{1}{n+1}$$

Case II. When m is a +ve integer and so in n :

$$I_{m,n} = \frac{m-1}{m+n}.\frac{m-3}{m+n-2}\ldots\frac{1}{n+2} I_{0,n}$$

Remarks

But $I_{0,n} = \int_0^{\pi/2} \cos^n x\, dx$

$$= \frac{n-1}{n}\frac{n-3}{n-2}\cdots\frac{1}{2}.\frac{\pi}{2} \qquad \text{[n is even]}$$

$$\therefore \quad I_{m,n} = \frac{m-1}{m+n}.\frac{m-3}{m+n-2}\cdots\frac{1}{n+2}.\frac{n-1}{n}.\frac{n-3}{n-2}\cdots\frac{1}{2}.\frac{\pi}{2}$$

Case III. When m is a +ve even integer but n is +ve odd integer :

$$I_{m,n} = \frac{m-1}{m+n}.\frac{m-3}{m+n-2}\cdots\frac{1}{n+2}I_{0,n}$$

But $I_{0,n} = \int_0^{\pi/2} \cos^n x dx = \frac{n-1}{2}\frac{n-3}{n-2}\cdots\frac{2}{3}$ [$\because$ n is odd integer]

$$\therefore \quad I_{m,n} = \frac{m-1}{m+n}.\frac{m-3}{m+n-2}\cdots\frac{1}{n+2}.\frac{n-1}{n}.\frac{n-3}{n-2}\cdots\frac{2}{3}$$

The value of given integral can be written by a simple rule as

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{[(m-1)\times \text{go on dim i. by } 2][(n-1)\times \text{go on dim i. by } 2}{(m+n)\times \text{go on dim i. by } 2}.\left(\frac{\pi}{2}\right)^*$$

$$\therefore \quad \int_0^{\pi/2} \sin^5 x \cos^4 x\, dx = \frac{4.3.2.1}{9.7.5.3} = \frac{8}{315}$$

[We do not write $\frac{\pi}{2}$ as both m and n are not even]

* Write $\frac{\pi}{2}$ if both m and n are +ve even integers otherwise don't write $\frac{\pi}{2}$.

Exercise 6.4

1. Connect $\int \sin^m x \cos^n x\, dx$ with $\int \sin^{m-2} x \cos^n x\, dx$.
2. Connect $\int \sin^m x \cos^n x\, dx$ with $\int \sin^{m+2} x \cos^n x\, dx$.
3. Prove that

   (i) $\int_0^{\pi/2} \sin^5 x \cos^6 x\, dx = \frac{8}{693}$ (ii) $\int_0^{\pi/2} \sin^6 x \cos^8 x\, dx = \frac{5\pi}{4096}$

4. Prove that

   (i) $\int_0^{\pi} \sin^6 \frac{\theta}{2} \cos^8 \frac{\theta}{2} d\theta = \frac{5\pi}{2^{11}}$ (ii) $\int_0^{2a} x^2 \sqrt{2ax - x^2}\, dx = \frac{5\pi}{8} a^4$.

Remarks

5. Evaluate the following integrals :

(i) $\int_0^\infty \frac{x^4}{(1+x^2)^4}dx$ (ii) $\int_0^{2a} x^{9/2}(2a-x)^{-1/2}dx$ (iii) $\int_0^{\pi/6} \cos^6 3x \sin^2 6x\,dx$

Ans. (i) $\frac{\pi}{32}$ (ii) $\frac{63}{8}\pi a^5$ (iii) $\frac{7\pi}{384}$.

Example 1. Connect $\int x^m(a+bx^n)^p dx$ with $\int x^m(a+bx^n)^{p-1}dx$.

Solution. Let P = x^{m+1} w.r.t. x, we have

Differentiating both sides w.r.t. x, we have

$$\frac{dP}{dx} = (m+1)x^m(a + bx^n)^p + x^{m+1}.p(a + bx^n)^{p-1}.bnx^{n-1}$$
$$= (m+1)x^m (a + bx^n)^p + pn\, bx^{m+n} (a + bx^n)^{p-1}$$
$$= (m+1)x^m (a + bx^n)^p + pn\, x^m.x^n (a + bx^n)^{p-1}$$
$$= (m+1)x^m (a + bx^n)^p + pnx^m (a + bx^n - a) (a + bx^n)^{p-1}$$
$$= (m+1)x^m (a + bx^n)^p + pnx^m (a + bx^n)^p - pnax^m (a + bx^n)^{p-1}$$
$$= (m+1+ pn)x^m (a + bx^n)^p - pnax^m (a + bx^n)^{p-1}$$

Integrating both sides w.r.t. x, we get

$$P = (m + 1 + pn) \int x^m(a+bx^n)^p dx - pna\int x^m(a+bx^n)^{p-1}dx$$

$\therefore$ $$(m + 1 + pn)\int x^m(a+bx^n)^p dx = \int x^m(a+bx^n)^{p-1}dx$$

Or $$\int x^m(a+bx^n)^p dx = \frac{x^{m+1}(a+bx^n)}{m+1+pn} + \frac{pna}{m+1+pn}\times\int x^m(a+bx^n)^{p-1}dx$$

Example 2. Obtain a reduction formula for $\int \frac{x^m dx}{(x^3-1)^{1/3}}$ and find the value of $\int x^8(x^3-1)^{-1/3}dx$**.**

Solution. Let us connect $\int x^m(x^3-1)^{-1/3}dx$ with $\int x^{m-3}(x^3-1)^{-1/3}dx$

Let $$P = x^{m-3+1}(x^3-1)^{-\frac{1}{3}+1}$$

i.e., $$P = x^{m-2} (x^3-1)^{2/3} \quad \ldots(1)$$

$\therefore$ $$\frac{dP}{dx} = x^{m-2}.\frac{2}{3}(x^3-1)^{-1/3} (3x^2) + (x^3-1)^{2/3} (m-2)x^{m-3}$$
$$= 2x^m(x^3-1)^{-1/3} + (m-2)x^{m-3}(x^3-1)^{2/3}$$
$$= 2x^m(x^3-1)^{-1/3} + (m-2)x^{m-3}(x^3-1)^{-1/3} (x^3-1)$$

Remarks

$= 2x^m(x^3 - 1)^{-1/3} + (m-2)x^m(x^3-1)^{-1/3} - (m-2)x^{m-3}(x^3-1)^{-1/3}$

$= mx^m(x^3-1)^{-1/3} - (m-2)x^{m-3}(x^3-1)^{-1/3}$

Integrating both sides w.r.t. x, we get

$$P = m\int x^m(x^3-1)^{-1/3}dx - (m-2)\int x^{m-3}(x^3-1)^{-1/3}dx$$

or $$m\int x^m(x^3-1)^{-1/3}dx = P + (m-2)\int x^{m-3}(x^3-1)^{-1/3}dx \qquad \ldots(2)$$

From (1) and (2), we get

$$\int x^m(x^3-1)^{-1/3}dx = \frac{x^{m-2}(x^3-1)^{2/3}}{m} + \frac{m-2}{m}\int x^{m-3}(x^3-1)^{-1/3}dx \qquad \ldots(3)$$

which is the required reduction formula.

Putting m = 8 in (3), we have

$$\int x^8(x^3-1)^{-1/3}dx = \frac{x^6(x^3-1)^{2/3}}{8} + \frac{6}{8}\int x^5(x^3-1)^{-1/3}dx \qquad \ldots(4)$$

Putting m = 5 in (3), we have

$$\int x^5(x^3-1)^{-1/3}dx = \frac{x^3(x^3-1)^{2/3}}{5} + \frac{3}{5}\int x^2(x^3-1)^{-1/3}dx \qquad \ldots(5)$$

But $$\int x^2(x^3-1)^{-1/3}dx = \frac{1}{3}(x^3-1)^{-1/3}(3x^2)dx = \frac{1}{3}\frac{(x^3-1)^{2/3}}{\frac{2}{3}} + c = \frac{1}{2}(x^3-1)^{2/3} + c_1$$

From (5), we get

$$\int x^5(x^3-1)^{-1/3}dx = \frac{x^3(x^3-1)^{2/3}}{5} + \frac{3}{5}\left[\frac{1}{2}(x^3-1)^{2/3} + c_1\right] = (x^3-1)^{2/3}\left[\frac{x^3}{5} + \frac{3}{10}\right] + c_2$$

Hence from (4), we get

$$\int x^8(x^3-1)^{-1/3}dx = \frac{x^6(x^3-1)^{2/3}}{8} + \frac{3}{4}\left[(x^3-1)^{2/3}\left(\frac{x^3}{5} + \frac{3}{10}\right) + c_2\right]$$

$$= (x^3-1)^{2/3}\left[\frac{x^6}{8} + \frac{3x^3}{20} + \frac{9}{40}\right] + C$$

Exercise 6.5

1. Connect $\int x^m(a+bx^n)^p dx$ with $\int x^{m+n}(a+bx^n)^p dx$.
2. Prove that $\int \frac{dx}{(a^2+x^2)^n} = \frac{x}{2a^2(n-1)(a^2+x^2)^{n-1}} + \frac{(2n-3)}{2a^2(n-1}\int \frac{dx}{(a^2+x^2)dx}$.
3. Obtain a reduction formula for $\int (x^2+a^2)^{n/2}\,dx$ and hence evaluate $\int (x^2+a^2)^{5/2}\,dx$.

Remarks

Ans. $\int (x^2+a^2)^{n/2}dx = \frac{x(x^2+a^2)^{n/2}}{n+1} + \frac{na^2}{n+1}\int (x^2+a^2)^{\frac{n}{2}-1}dx$ and

$$\int (x^2+a^2)^{5/2}dx = \frac{x(x^2+a^2)^{5/2}}{6} + \frac{5a^2}{24}x(x^2+a^2)^{3/2} + \frac{5a^4}{16}x\sqrt{x^2+a^2}$$

$$+\frac{5}{16}a^6 \log(x+\sqrt{x^2+a^2}).$$

4. If $I_{m,n} = \int_0^1 x^m(1-x)^n dx$, prove that $(m + n + 1)\, I_{m,n} = n\, I_{m,n-1}$.

   Hence find the value of $\int_0^1 x^4(1-x)^3 dx$.

Ans. $\frac{1}{280}$.

5. If $I_{m,n} = \int \frac{x^m}{(x^2+1)^n}dx$, then prove that

$$2(n-1)I_{m,n} = -x^{m-1}(x^2+1)^{-(n-1)} + (m-1)I_{m-2,n-1}.$$

6. Hence evaluate $\int_0^{2a} x^m\sqrt{2ax-x^2}dx$ then $\int_0^{2a} x^3\sqrt{2ax-x^2}dx$.

Ans. $\int x^m\sqrt{2ax-x^2}dx = \frac{x^{m-1}(2ax-x^2)^{3/2}}{m+2} + \frac{a(2m+1)}{m+2}\int x^{m-1}\sqrt{2ax-x^2}dx$

$$I_m = \frac{\pi a^{m+2}(2m+1)!}{2^m m!(m+2)!};\ I_3 = \frac{7}{8}\pi a^5.$$

Keywords : Reduction, Connection.

Summary : Firstly we define the reduction formulae then we have the reduction formulae for $\int \sin^n x dx$, $\int \tan^n x dx$, $\int \sec^n x dx$ etc. and based examples & exercise. Solution of integrals by method of connect and evaluation of definite integrals.

Remarks

CHAPTER – VII

RECTIFICATION

7.0 STRUCTURE

7.1 INTRODUCTION

If a locus is given in the form of a function $y = f(x)$, $x_0 \leq x \leq x_n$ and if f is continuous in this interval, then locus is called an arc. While discussing the length of arc, we use the word "length" in place of the words "measure of length". Thus length of an arc is a number without any units of measurements attached to it. We have derivation of intrinsic equation of a curve from different types of equations.

7.2 OBJECTIVE

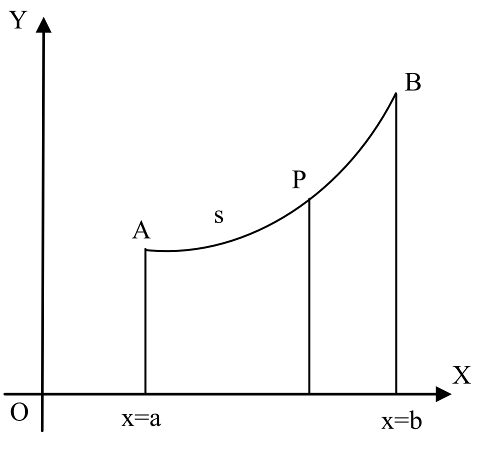
After reading this lesion you must be able to

- Understand about fundamental theorem
- Length of an arc.
- Understand the meaning and derivation of intrinsic equation from the different types of curves.

7.1 FUNDAMENTAL THEOREM ABOUT RECTIFICATION

If f(x) is such that f'(x) exists and is continuous in the interval [a, b] then the curve y = f(x) is rectifiable in the internal and its arc length s is given by

$$s = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$



Let AB be the part of the graph of y = f(x) for $a \leq x \leq b$. Let s denote the length of the curve measured from the fixed point A to a variable point P(x, y) of the curve.

From differential calculus, we know that

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Remarks

Integrating both sides w.r.t. x from x = a and x = b, we have

$$\int_0^b \frac{ds}{dx}dx = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

or $$[s]_{x=b} - [s]_{x=a} = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$\Rightarrow$ $$(\text{Arc AB}) - (0) = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

[$\because$ Length is being measured from fixed point A, so s corresponding t x = a is zero]

Hence $$\text{arc AB} = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \qquad \ldots(1)$$

Remark 1. If the curve is defined by x = f(y), where f'(y) exists and is continuous on [c, d], then the length of the curve is given by

$$\int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy \qquad \ldots(2)$$

To derive the above result, we use

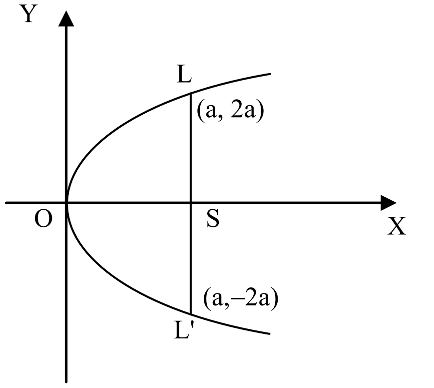
$$\frac{ds}{dy} = \sqrt{1+\left(\frac{dx}{dy}\right)^2}$$

2. While using formula (1), y must be expressed as an explicit function of x and $\frac{dy}{dx}$ must be a function of x only.

If we use formula (2), then x must be expressed as an explicit function of y and $\frac{dx}{dy}$ must also be a function of y only.

7.2. WORKING RULE TO FIND THE LENGTH OF AN ARC OF A CURVE, WHEN ITS EQUATION IS IN CARTESIAN FORM [y = f(x)].

1. If the end points of the arc, whose length has to be found, are given, then we know the limits of integration. In such case we need not trace the curve. However in problems where the limits of integration are not given, we have to trace the curve roughly, to find the limits of integration.



2. Express y as an explicit function of x and then find $\frac{dy}{dx}$.

3. Get the required arc length by using the formula $\int_0^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$, where a, b are the limits of integration obtained in step 1.

Example 1. Find the length of the arc of the parabola $y^2 = 4ax$ from the vertex to an extremity of the latus rectum.

Remarks

Solution. Let O be the vertex and L, an extremity of the latus rectum of the given parabola

$$y^2 = 4ax$$

or
$$x = \frac{y^2}{4a}$$

Differentiating w.r.t y,

$$\frac{dx}{dy} = \frac{2y}{4a} = \frac{y}{2a} \; .$$

To find the length of the arc from O(0, 0) to L(a, 2a); we see that y varies from 0 to 2a.

∴ Required length of the arc

$$\int_0^{2a}\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy = \int_0^{2a}\sqrt{1+\left(\frac{y}{2a}\right)^2}\,dy \quad = \frac{1}{2a}\int_0^{2a}\sqrt{(y)^2+(2a)^2}\,dy$$

$$= \frac{1}{2a}\left[\frac{y}{2}\sqrt{y^2+4a^2} . \frac{4a^2}{2}\log\frac{y+\sqrt{y^2+4a^2}}{2a}\right]_0^{2a}$$

$$= \frac{1}{2a}\left[\frac{2a}{2}\sqrt{4a^2+4a^2}+2a^2\log\frac{2a\sqrt{4a^2+4a^2}}{2a}-0\right]$$

$$= [a\sqrt{2} + a\log(1+\sqrt{2})] \qquad \text{[On simplification]}$$

$$= a[\sqrt{2} + \log(\sqrt{2}+1)]$$

Example 2. Show that the length of the curve $x^2(a^2 - x^2) = 8a^2y^2$ is $\pi a\sqrt{2}$.

Solution. The given curve is $8a^2y^2 = x^2(a^2 - x^2)$ …(1)

Let us trace the curve roughly to find the limits of integration.

1. The curve is symmetrical about x–axis and y–axis.
2. The curve passes through the origin and the tangents at the origin are given by $y = \pm\frac{1}{2\sqrt{2}}x$

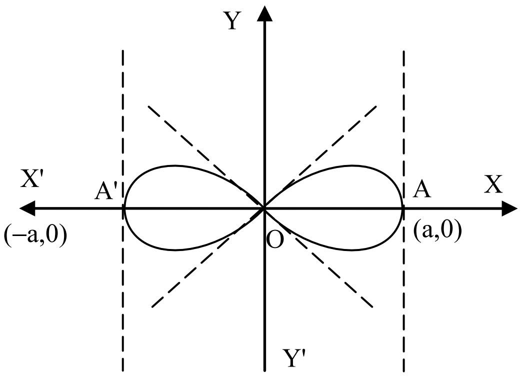
The real and distinct tangents show that origin is a node.

3. If $y = 0$, $x^2(a^2 - x^2) = 0 \;\Rightarrow\; x = 0, \pm a$.

Therefore the curve meets the x–axis at (0,0), (a,0), (–a,0).

If x = 0, y = 0 i.e., the curve meets the y–axis at the origin only.

4. The curve has no asymptotes.
5. In the curve $8a^2y^2 = x^2(a^2 - x^2)$, L.H.S. is always positive.



∴ R.H.S. must also be positive

i.e., $(a^2 - x^2) \geq 0$

or $x^2 \leq a^2$

∴ $-a \leq x \leq a$

Thus the whole curve lies between the lines x = –a and x = a. The curve consists of two symmetrical loops as shown in fig.

Remarks

From (1), $y = \frac{1}{\sqrt{8a^2}} x\sqrt{a^2 - x^2}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{8a^2}}\left[x\left(\frac{-2x}{2\sqrt{a^2-x^2}}\right)+\sqrt{a^2-x^2}\right]$$

$$= \frac{1}{\sqrt{8a^2}}\left[\frac{a^2-2x^2}{\sqrt{a^2-x^2}}\right]$$

$\therefore$ Required length = 4(arc OA)

$$= 4\int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 4\int_0^a \sqrt{1+\frac{(a^2-2x^2)^2}{8a^2(a^2-x^2)}}\,dx$$

$$= 4\int_0^a \sqrt{\frac{8a^4-8a^2x^2+a^4+4x^4-4a^2x^2}{8a^2(a^2-x^2)}}\,dx$$

$$= \frac{4}{\sqrt{8a^2}}\int_0^a \sqrt{\frac{9a^4-12a^2x^2+4x^4}{a^2-x^2}}$$

$$= \frac{4}{2a\sqrt{2}}\int_0^a \frac{3a^2-2x^2}{\sqrt{a^2-x^2}}\,dx = \frac{\sqrt{2}}{a}\int_0^a \frac{2(a^2-x^2)+a^2}{\sqrt{a^2-x^2}}\,dx$$

$$= \frac{\sqrt{2}}{a}\int_0^a \left(2\sqrt{a^2-x^2}+\frac{a^2}{\sqrt{a^2-x^2}}\right)dx$$

$$= \frac{\sqrt{2}}{a}\left|2\left(x\frac{\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right)+a^2\sin^{-1}\frac{x}{a}\right|_0^a$$

$$= \frac{\sqrt{2}}{a}\left|x\sqrt{a^2-x^2}+2a^2\sin^{-1}\frac{x}{a}\right|_0^a$$

$$= \frac{\sqrt{2}}{a}\,|\,2a^2\sin^{-1}1-\sin^{-1}0\,| = \frac{\sqrt{2}}{a}.2a^2.\frac{\pi}{2} = \pi a\sqrt{2}\ .$$

Example 3. Find the perimeter of the loop of the curve $9ay^2 = (x - 2a)\,(x - 5a)^2$.

Solution. The given curve is $9ay^2 = (x- 2a)(x - 5a)^3$ …(1)

Let us trace the curve roughly to find the limits of integration.

1. The curve is symmetrical about x–axis
2. The curve does not pass through the origin.
3. If $y = 0$, $x = 2a, 5a$.

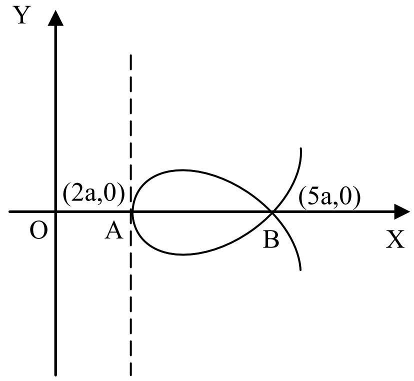
Thus the curve meets the x–axis at (2a, 0), (5a, 0). Shifting the origin to (2a, 0), we find the tangent is parallel to new y–axis.

Remarks

Also shifting the origin to (5a, 0), we see that tangents at the new origin are $y = \pm\frac{1}{\sqrt{3}}x$

$\therefore$ (5a, 0) is a node.

So, there is a loop between x = 2a and x = 5a.



4. The curve has no asymptotes.

5. In the curve $9ay^2 = (x - 2a)(x - 5a)^2$, L.H.S. is always positive. This happens when x > 2a

Also as x increases from 5a to ∞, y also increases from 0 to ∞.

The shape of the curve is shows in fig.

For the loops, x varies from 2a to 5a and perimeter of loops = 2 × are of loop above x–axis.

Also from (1), $y = \frac{1}{3\sqrt{a}}(x-5a)\sqrt{x-2a}$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{3\sqrt{a}}\left[\sqrt{x-2a} + \frac{x-5a}{2\sqrt{x-2a}}\right] =$$

$$\frac{1}{3\sqrt{a}}\left[\frac{2(x-2a)+x-5a}{2\sqrt{x-2a}}\right] = \frac{3x-9a}{6\sqrt{a}\sqrt{x-2a}} = \frac{x-3a}{2\sqrt{a}\sqrt{x-2a}}$$

Required perimeter = 2 × arc AB

$$= 2\int_{2a}^{5a}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 2\int\sqrt{1+\frac{(x-3a)^2}{4a(x-2a)}}dx$$

$$= 2\int_{2a}^{5a}\sqrt{\frac{4ax-8a^2+x^2+9a^2-6ax}{4a(x-2a)}}dx$$

$$= 2\int_{2a}^{5a}\frac{x-a}{2\sqrt{a}\sqrt{x-2a}}dx = \frac{1}{\sqrt{a}}\int_{2a}^{5a}\frac{x-2a+a}{\sqrt{x-2a}}dx$$

$$= \frac{1}{\sqrt{a}}\int_{2a}^{5a}[(x-2a)^{1/2}+a(x-2a)^{-1/2}]dx$$

$$= \frac{1}{\sqrt{a}}\left[\frac{2}{3}(x-2a)^{3/2}+2a(x-2a)^{1/2}\right]_{2a}^{5a}$$

$$= \frac{1}{\sqrt{a}}\left[\frac{2}{3}(3a)^{3/2}+2a(3a)^{1/2}\right] = 4\sqrt{3}a\ .$$

$$= 2\int_{2a}^{5a}\frac{x-a}{2\sqrt{a}\sqrt{x-2a}}dx = \frac{1}{\sqrt{a}}\int_{2a}^{5a}\frac{x-2a+a}{\sqrt{x-2a}}dx$$

Remarks

$$= \frac{1}{\sqrt{a}} \int_{2a}^{5a} [(x-2a)^{1/2} + a(x-2a)^{-1/2}]dx$$

$$= \frac{1}{\sqrt{a}} \left[\frac{2}{3}(x-2a)^{3/2} + 2a(x-2a)^{1/2} \right]_{2a}^{5a}$$

$$= \frac{1}{\sqrt{a}} \left[\frac{2}{3}(3a)^{3/2} + 2a(3a)^{1/2} \right] = 4\sqrt{3}\,a\,.$$

Example 4. Show that in the cantenary $y = \cosh \dfrac{x}{c}$, the length of the arc from the vertex (0, c) to any point (x, y) is given by

Solution. The given equation of the curve is

$$y = \cosh \frac{x}{c}$$

$$\therefore \qquad \frac{dy}{dx} = c \sinh \frac{x}{c} . \frac{1}{c} = \sinh \frac{x}{c}$$

(i) Required length,

$$s = \int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^x \sqrt{1 + \sinh^2 \frac{x}{c}}\, dx$$

$$= \int_0^x \cosh \frac{x}{c} dx = c\left[\sinh \frac{x}{c}\right]_0^x$$

$$= c\left(\sinh \frac{x}{c} - 0\right) = c \sinh \frac{x}{c}$$

(ii) Also

$$s^2 = c^2 \sinh^2 \frac{x}{c} = c^2\left[\cos h^2 \frac{x}{c} - 1\right] = c^2\left[\left(\frac{y}{c}\right)^2 - 1\right] = y^2 - c^2\,.$$

Exercise 7.1

1. Find length of the arc $x^2 + y^2 - 2ax = 0$ in first quadrant.

Ans. 9π.

2. Show that the length of the loop of the curve $3ay^2 = x(x-a)^2$ is $\dfrac{49}{\sqrt{3}}$.

3. Find the length of the boundary of the region bounded by the curve $y = \dfrac{1}{2}x^2 + 1$ and the line y = x, x = 0 and x = 2.

Ans. $2\sqrt{2} + 2 + \sqrt{5} + \dfrac{1}{2}\sinh^{-1} 2$

4. Find the length of an arc of parabola $x^2 = 4ay$.

(i) From the vertex to an extremity of the latus rectum.

(ii) Cut off by latus rectum.

Ans. (i) $a[\sqrt{2} + \log(\sqrt{2}+1)]$ (ii) $2a[\sqrt{2} + \log(\sqrt{2}+1)]$

Remarks

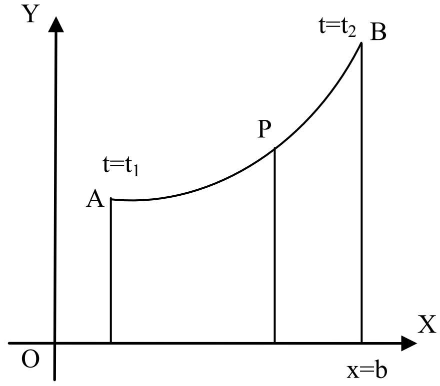
5. Find the length of the boundary of the region bounded by the curve $y = x^2 + 1$, and lines $y = x$, $x = 0$ and $y = 2$.

Ans. $2+2\sqrt{2}+\frac{\sqrt{5}}{2}+\frac{1}{4}\log(2+\sqrt{5})$

7.3 LENGTH OF THE PARAMETRIC CURVES

If the co-ordinates of a point (x, y) on the curve are given by $x = f(t)$ and $y = g(t)$ and if $f'(t)$ and $g'(t)$ exist and are continuous in the interval $[t_1, t_2]$, then the arc length between the points where $t = t_1$ to $t = t_2$ is given by

$$\int_{t_1}^{t_2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$



Example 1. Find the whole length of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$.

Solution. The equation of the curve is $x^{2/3} + y^{2/3} = a^{2/3}$. …(1)

Its parametric equations are $x = a\cos^3\theta$, $y = a\sin^3\theta$ …(2)

From (2), we see that x and y cannot be numerically greater than a.

From (1), we see that

(i) The curve is symmetrical about x-axis, about y-axis and about the line $y = x$.

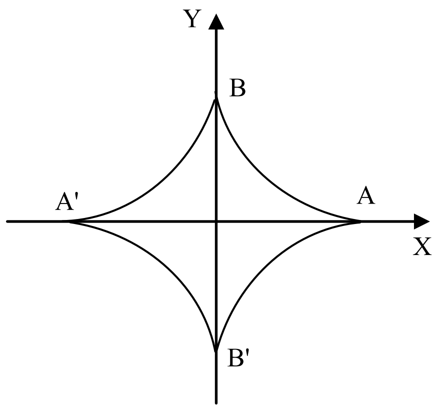
(ii) The curve meets the x-axis at $(\pm a, 0)$

(iii) The curve meets the y-axis at $(0, \pm a)$

Again from (1), we have

$$\frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}\frac{dy}{dx}=0$$

$$\therefore \qquad \frac{dy}{dx}=-\left(\frac{y}{x}\right)^{1/3}$$



In the first quadrant $\frac{dy}{dx}$ is –ve, which shows that in the first quadrant y decreases as x increases. The shape of the curve is as shows in fig. and for the curve in the first quadrant, θ varies from 0 to $\frac{\pi}{2}$.

From (2), $\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$

and $\frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$

$$\therefore \qquad \left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2 = 9a^2\cos^4\theta.\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta$$

$$= 9a^2\sin^2\theta.\cos^2\theta(\cos^2\theta+\sin^2\theta) = 9a^2\sin^2\theta\cos^2\theta$$

∴ Required whole length of the curve = 4 × length of curve in the first quadrant

Remarks

$$= 4\int_0^{\pi/2}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta = 4\int_0^{\pi/2} 3a\sin\theta\cos\theta d\theta$$

$$= 12a\int_0^{\pi/2}\sin\theta\cos\theta d\theta = 6a\int_0^{\pi/2}\sin 2\theta d\theta$$

$$= -\frac{6a}{2}[\cos 2\theta]_0^{\pi/2}$$

$$= -3a\,[\cos\pi - \cos 0] = -3a(-1-1) = 6a.$$

Example 2. (i) Find the length of the arc of the curve $x = e^{\theta}\sin\theta$, $y = e^{\theta}\cos\theta$ from $\theta = 0$ to $\theta = \frac{\pi}{2}$.

(ii) Find the length of the arc of the curve

$$x = e^{\theta}\left(\sin\frac{\theta}{2}+2\cos\frac{\theta}{2}\right),\ y = e^{\theta}\left(\cos\frac{\theta}{2}-2\sin\frac{\theta}{2}\right)$$

measured from $\theta = 0$ to $\theta = \pi$.

Solution. (i) The given curve is

$x = e^{\theta}\sin\theta$, $y = e^{\theta}\cos\theta$

$$\therefore \quad \frac{dx}{d\theta} = e^{\theta}(\cos\theta+\sin\theta)$$

And
$$\frac{dy}{d\theta} = e^{\theta}(\cos\theta-\sin\theta)$$

$$\therefore \quad \left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2 = e^{2\theta}[(\cos\theta+\sin\theta)^2+(\cos\theta-\sin\theta)^2]$$

$$= e^{2\theta}.\,2(\cos^2\theta+\sin^2\theta) = 2e^{2\theta}$$

$$\therefore \quad \text{Required length} = \int_0^{\pi/2}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

$$= \int_0^{\pi/2}\sqrt{2e^{2\theta}} = \sqrt{2}\int_0^{\pi/2} e^{\theta}d\theta$$

$$= \sqrt{2}[e^{\theta}]_0^{\pi/2} = \sqrt{2}[e^{\pi/2}-e^{0}] = \sqrt{2}(e^{\pi/2}-1).$$

(ii) The given curve is $x = e^{\theta}\left(\sin\frac{\theta}{2}+2\cos\frac{\theta}{2}\right)$, $y = e^{\theta}\left(\cos\frac{\theta}{2}-2\sin\frac{\theta}{2}\right)$

$$\therefore \quad \frac{dx}{d\theta} = e^{\theta}\left(\frac{1}{2}\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)+e^{\theta}\left(\sin\frac{\theta}{2}+2\cos\frac{\theta}{2}\right)$$

$$= \frac{5}{2}e^{\theta}\cos\frac{\theta}{2}$$

and
$$\frac{dy}{d\theta} = e^{\theta}\left(-\frac{1}{2}\sin\frac{\theta}{2}-\cos\frac{\theta}{2}\right)+e^{\theta}\left(\cos\frac{\theta}{2}-2\sin\frac{\theta}{2}\right)$$

$$= -\frac{5}{2}e^{\theta}\sin\frac{\theta}{2}$$

Remarks

$$\therefore \left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2=\frac{25}{4}e^{2\theta}\left(\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}\right)=\frac{25}{4}e^{2\theta}$$

$$\therefore \quad \text{Required length} = \int_0^{\pi}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

$$= \int_0^{\pi}\sqrt{\frac{25}{4}e^{2\theta}}\,d\theta=\int_0^{\pi}\frac{5}{2}e^{\theta}d\theta$$

$$= \frac{5}{2}[e^{\theta}]_0^{\pi}=\frac{5}{2}(e^{\pi}-1)\ .$$

Example 5. Prove that the loop of the curve $x = t^2,\ y = t-\frac{1}{3}t^3$ is of length $4\sqrt{3}$.

Solution. The given curve is $x = t^2,\ y = t-\frac{1}{3}t^3$ …(1)

Let us make a rough sketch of the curve to get the limits of integration.

(i) As x is even function of t and y is odd function of t, the curve is symmetrical about x-axis.

(ii) If t = 0, then x = 0 and y = 0 i.e., the curve passes through the origin.

(iii) For y = 0, from (1), $t = 0, \pm\sqrt{3}$

If t = 0, x = 0 and if $t = \pm\sqrt{3}$, x = 3

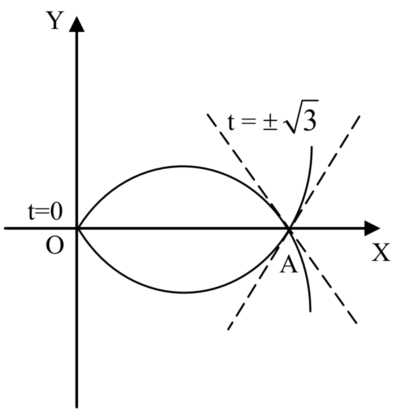
Thus the curve meets x-axis at (0, 0), (3, 0)

Also, for x = 0, t = 0and so y = 0

Thus the curve meets y-axis at the origin (0,0)

(iv) If $x \to\infty$, then $t \to\infty$ and $y \to-\infty$

The curve has no asymptotes but it goes upto ∞.



(v) From (1), $\frac{dx}{dt}=2t,\ \frac{dy}{dt}=1-t^2$

$$\therefore \quad \frac{dy}{dx}=\frac{1-t^2}{2t}$$

Now $\frac{dy}{dx}$ = 0 if $1 - t^2 = 0$ $\Rightarrow$ $t = \pm 1$

When $t = 1,\ x = 1,\ y = \frac{2}{3}$

When $t = -1,\ x = 1,\ y = -\frac{2}{3}$

$\therefore$ The tangent is parallel to x-axis at the points

$$\left(1,\frac{2}{3}\right)\text{and}\left(1,-\frac{2}{3}\right)$$

Also $\frac{dy}{dx}\to\infty$ when $t \to 0$

When $t \to 0$, then x and y both $\to 0$

Thus the tangent is perpendicular to x-axis at (0, 0)

Remarks

When t = $\pm\sqrt{3}$, $\quad \frac{dy}{dx} = \mp\frac{1}{\sqrt{3}}$

which shows that at t = $\pm\sqrt{3}$ (x = 3, y = 0) there are two tangents inclined at an angle $\mp 30°$ with x-axis. Also t is imaginary if x < 0.

So, no portion of the curve lies to left y-axis. The shape of the curve is shows in fig. For the upper half of the loop, t varies from 0 to $\sqrt{3}$.

∴ Required length of the loop = 2[length of upper half of loop]

$$= 2\int_0^{\sqrt{3}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= 2\int_0^{\sqrt{3}} \sqrt{(2t)^2 + (1-t^2)^2}\, dt = 2\int_0^{\sqrt{3}} (1+t^2)\, dt$$

$$= 2\left|t + \frac{t^3}{3}\right|_0^{\sqrt{3}} = 2\left[(\sqrt{3}-0) + \left(\frac{3\sqrt{3}}{3} - 0\right)\right]$$

$$= 2(\sqrt{3}+\sqrt{3}) = 4\sqrt{3}.$$

Example 4. Find the length of one arch of the cycloid x = a(θ–sin θ), y = a(1– cos θ).

Solution. The equations of the cycloid are

$$x = a(\theta - \sin\theta),\ y = a(1-\cos\theta) \qquad \ldots(1)$$

Let us make a rough sketch of the curve to find the limits of integration.

(i) The curve is symmetrical about y-axis because x is an odd function of θ and y is an even function of θ.

(ii) For x = 0, θ– sin θ = 0 ⇒θ = 0

Also θ = 0 ⇒ y = 0

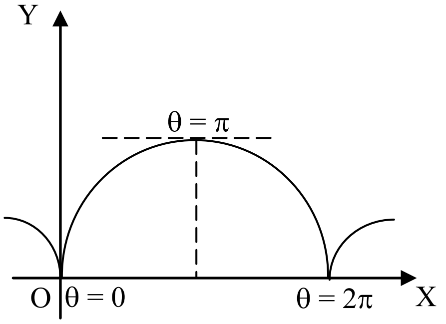
So, the curve passes through the origin.

(iii) From (1), $\frac{dx}{d\theta} = a(1-\cos\theta)$

$\frac{dy}{d\theta} = a\sin\theta$

∴ $\frac{dy}{dx} = \frac{a\sin\theta}{a(1-\cos\theta)}$

$= \cot\frac{\theta}{2}$



The corresponding values of x, y and $\frac{dy}{dx}$ for various values of θ are as given below :

θ :	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
x :	0	$a\left(\frac{\pi}{2}-1\right)$	aπ	$a\left(\frac{3\pi}{2}+1\right)$	2aπ
y :	0	a	2a	a	0
$\frac{dy}{dx}$:	∞	1	0	–1	∞

Remarks The figure shows one arch of the curve

For-one complete arch of the curve, θ varies from 0 to 2π.

∴ Required length of one arch = $\int_0^{2\pi}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$

$$= a\int_0^{2\pi}\sqrt{a^2(1-\cos\theta)^2}\,d\theta = 2a\int_0^{2\pi}\sin\frac{\theta}{2}\,d\theta$$

$$= 2a\left[\frac{-\cos\frac{\theta}{2}}{\frac{1}{2}}\right]_0^{2\pi} = -4a[\cos\pi-\cos 0]$$

$$= -4a(-1-1) = 8a.$$

Exercise 7.2

1. Rectify the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.
2. Find length of the arc in the first quadrant of the curve $\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}=1$.

Ans. $\frac{a^2+ab+b^2}{a+b}$.

3. Find length of the complete cycloid given by

   x = a (θ + sin θ), y = a(1 + cos θ)

Ans. 8a.

4. Show that the length of the arc of the curve

   x = a sin 2θ(1 + cos 2θ),y = a cos 2θ(1 – cos 2θ)

measured from origin to any point (x, y) is $\frac{4a}{3}\sin 3\theta$.

5. Rectify the cycloid x = a(θ– sin θ), y = a(1 + cos θ)

[Hint : Ref. solved example 4]

Ans. 8a.

7.4 LENGTH OF THE POLAR CURVES

If f(θ) and f'(θ) are continuous on a closed interval [α, β], then the length s of arc of the curve r = f(θ), between the points for which θ = α to θ = β is given by

$$\int_\alpha^\beta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

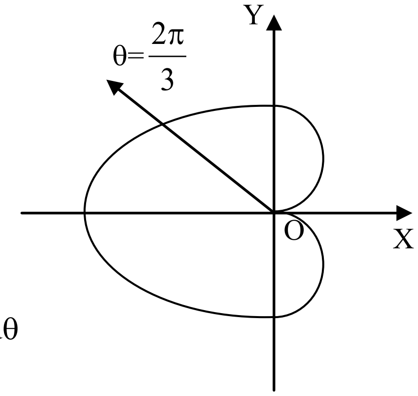
Cor. If the equation of the curve is θ = f(r), then the length of arc between two points whose radii vectors are r_1 and r_2 is

$$s = \int_{r_1}^{r_2}\sqrt{1+\left(r\frac{d\theta}{dr}\right)^2}\,dr$$

Remarks

Example 1. Find the whole length of the curve r = a(1 – cos θ) and show that the arc of the upper half of the curve is bisected by $\theta = \dfrac{2\pi}{3}$.

Solution. The equation of the curve is



$$r = a\,(1 - \cos\theta)$$

$$\therefore \qquad \frac{dr}{d\theta} = a \sin\theta$$

Thus the whole length of the curve is

$$= 2\int_0^{\pi} \sqrt{a^2 \sin^2\theta + a^2 (1-\cos\theta)^2}\, d\theta$$

$$= 2a\int_0^{\pi} \sqrt{1 - 2\cos\theta + \sin^2\theta + \cos^2\theta}\, d\theta$$

$$= 2a\int_0^{\pi} \sqrt{2 - 2\cos\theta}\, d\theta = 4a\int_0^{\pi} \sin\frac{\theta}{2}\, d\theta$$

$$= 4a \left| \frac{-\cos\frac{\theta}{2}}{\frac{1}{2}} \right|_0^{\pi} = -8a[0 - 1] = 8a.$$

Length of the upper half = 4a

Length of the curve from θ = 0 to $\theta = \dfrac{2\pi}{3}$.

$$= \int_0^{2\pi/3} \sqrt{a^2 \sin^2\theta + a^2 (1-\cos\theta)^2}\, d\theta$$

$$= 2a\int_0^{2\pi/3} \sin\frac{\theta}{2}\, d\theta = 2a \left| \frac{-\cos\frac{\theta}{2}}{\frac{1}{2}} \right|_0^{2\pi/3}$$

$$= -4a\left[\cos\frac{\pi}{3} - \cos 0\right] = -4a\left(\frac{1}{2} - 1\right) = 2a$$

$$= \frac{1}{2}\text{(length of the upper half)}$$

which shows that the upper half of the curve is bisected at $\theta = \dfrac{2\pi}{3}$.

Example 2. Find the length of the arc of the parabola $\dfrac{2a}{r} = 1 + \cos\theta$ cut off by its latus rectum.

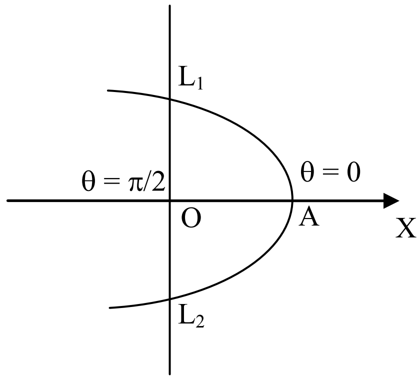
Solution. The equation of the parabola is

$$\frac{2a}{r} = 1 + \cos\theta$$

Remarks

Or $$r = \frac{2a}{1+\cos\theta} = a\sec^2\frac{\theta}{2} \qquad \ldots(1)$$

1. It represents a parabola with focus at the pole and initial line OX as the axis of parabola.
2. It is symmetrical about the initial line. The upper half of the curve intercepted by the latus rectum L_1OL_2 is AL_1.



For this arc, θ varies from 0 to $\frac{\pi}{2}$.

From (1),

$$\frac{dr}{d\theta} = \left(2a\sec\frac{\theta}{2}\right)\left(\sec\frac{\theta}{2}.\tan\frac{\theta}{2}\right)\left(\frac{1}{2}\right) = a\sec^2\frac{\theta}{2}\tan\frac{\theta}{2}$$

∴ Required length of arc of parabola cut off by its latus rectum

$$= \text{Length of arc } L_2AL_1$$
$$= 2(\text{arc } AL_1)$$
$$= 2\int_0^{\pi/2}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$
$$= 2\int_0^{\pi/2}\sqrt{a^2\sec^4\frac{\theta}{2}+a^2\sec^4\frac{\theta}{2}\tan^2\frac{\theta}{2}}\,d\theta$$
$$= 2a\int_0^{\pi/2}\sec^3\frac{\theta}{2}\,d\theta \qquad \text{[See note after the solution]}$$
$$= 2a\left[\sec\frac{\theta}{2}\tan\frac{\theta}{2}+\log\left(\sec\frac{\theta}{2}+\tan\frac{\theta}{2}\right)\right]_0^{\pi/2}$$
$$= 2a[\sqrt{2}+\log(\sqrt{2}+1)-\log(1+0)]$$
$$= 2a[\sqrt{2}+\log(\sqrt{2}+1)]$$

Note. The value of $\int\sec^3\frac{\theta}{2}d\theta$ can be found by using reduction formula

$$\int\sec^n x\,dx = \frac{\sec^{n-2}x\tan x}{n-1}+\frac{n-2}{n+1}\int\sec^{n-2}x\,dx$$

or alternatively integrating by parts using

$$\int\sec^3\frac{\theta}{2}d\theta = \int\sec\frac{\theta}{2}.\sec^2\frac{\theta}{2}d\theta.$$

Exercise 7.3

1. Find the entire length of the cardioid r = a(1 + cos θ) and show that the arc of the upper half is bisected by $\theta = \frac{\pi}{3}$.
2. Find the length of a loop of the curve $r = a(\theta^2 - 1)$.
3. Find the length of the arc of cardioid r = a(1 − cos θ) between the points whose vectorial angles are α and β.

Ans. $4a\left(\cos\frac{\alpha}{2}-\cos\frac{\beta}{2}\right)$

Remarks

4. Show that the arc of the hyperbolic spiral $r\theta = a$ taken from the point $r = a$ to $r = 2a$ is $a\left[\sqrt{5}-\sqrt{2}+\log\left(\frac{2+\sqrt{8}}{1+\sqrt{5}}\right)\right]$.

5. Find the length of the first spiral of the curve $r = ae^{m\theta}$.

7.5 To prove that the length of the arc of the curve p = f(r) between the points r = a and r = b is

$$\int_a^b \frac{r}{\sqrt{r^2-p^2}}\,dr$$

Example 1. Prove that the length of the arc of the curve $p = r \sin \beta$ between the limits r_1 and r_2 is $(r_2 - r_1) \sec \beta$.

Solution. The required length of the curve

$$= \int_{r_1}^{r_2} \frac{r}{\sqrt{r^2-p^2}}\,dr = \int_{r_1}^{r_2} \frac{r}{\sqrt{r^2-r^2\sin^2\beta}}\,dr$$

$$= \int_{r_1}^{r_2} \frac{dr}{\cos\beta} = \sec\beta\int_{r_1}^{r_2} 1.dr$$

$= (r_2 - r_1) \sec \beta$.

Example 2. Show that the length of the arc of the hyperbola $xy = a^2$ between the limits $x = b$ and $x = c$ is equal to the arc of the curve $p^2(a^4 + r^4) = a^4r^2$ between the limits $r = b$, $r = c$.

Solution. The hyperbola is $y = \frac{a^2}{x}$

$$\therefore \quad \frac{dy}{dx} = -\frac{a^2}{x^2}$$

$\therefore$ Length of the arc of the hyperbola $= \int_b^c \sqrt{1+\frac{a^4}{x^4}}\,dx = \int_b^c \frac{\sqrt{x^4+a^4}}{x^2}\,dx \qquad \ldots(1)$

For the curve $p^2(a^4 + r^4) = a^4r^2$

i.e., $p^2 = \frac{a^4r^2}{a^4+r^4}$

$\therefore \quad r^2 - p^2 = r^2 - \frac{a^4r^2}{a^4+r^4} = \frac{r^2(a^4+r^4)-a^4r^2}{a^4+r^4} = \frac{r^6}{a^4+r^4}$

$\therefore$ Length of the arc $= \int_b^c \frac{r}{\sqrt{r^2-p^2}}\,dr = \int_b^c \frac{r}{\frac{r^3}{\sqrt{a^4+r^4}}}\,dr$

$$= \int_b^c \frac{\sqrt{a^4+r^4}}{r^2}\,dr \qquad \ldots(2)$$

Since integrals (1) and (2) are equal, therefore the two lengths are also equal.

7.6. INTRINSIC EQUATION OF A CURVE

Definition. If s denotes the length of the arc of a curve measured from some fixed point A to a variable point P and ψ is the angle between the tangents at P and A (or any other fixed line through A) then the relation between a and ψ is called an **intrinsic equation** of the curve.

Here s and ψ are called the intrinsic co-ordinates.

Remarks

7.7. DERIVATION OF THE INTRINSIC EQUATION OF A CURVE FROM THE CARTESIAN EQUATION

Let the Cartesian equation of the curve be $y = f(x)$

Let one curve passes through O which we take as the fixed point from where the arc OP = s is measured and x-axis be the tangent to the curve at O.

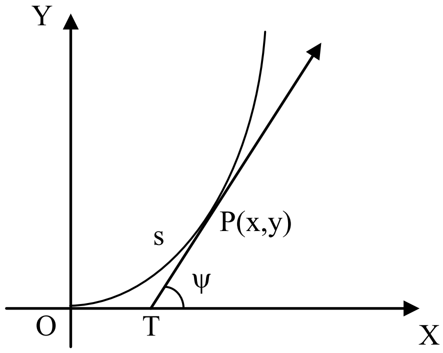
If ψ = angle between the tangent at P and the x-axis, then

$$\tan\psi = \frac{dy}{dx} = f'(x) \qquad \ldots(2)$$

and

$$s = \int_0^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_0^x \sqrt{1+[f'(x)]^2}\,dx \qquad \ldots(3)$$



Eliminating x from (2) and (3), we obtain the required intrinsic equation.

7.8. DERIVATION OF THE INTRINSIC EQUATION OF A CURVE FROM THE PARAMETRIC EQUATIONS

Let the parametric equations of the curve be

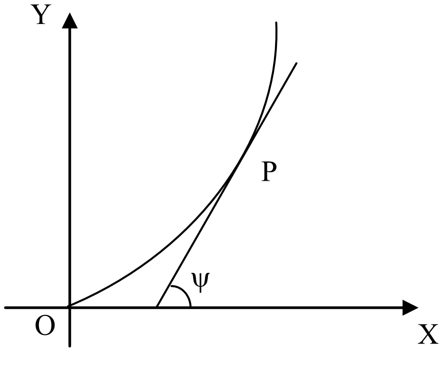
$$x = f(t), \quad y = g(t) \qquad \ldots(1)$$

Let the curve passes through O which we take as the fixed point from where the arc OP = s is the measured and x-axis be the tangent to the curve at O.

Let $t = 0$ at O.

If ψ = angle between the tangent at P(x, y) and the x-axis, then

$$\tan\psi = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)} \qquad \ldots(2)$$



and

$$s = \int_0^t \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt = \int_0^t \sqrt{[f'(t)]^2+[g'(t)]^2}\,dt \qquad \ldots(3)$$

Eliminating t from (2) and (3), we obtain the required intrinsic equation.

Remarks

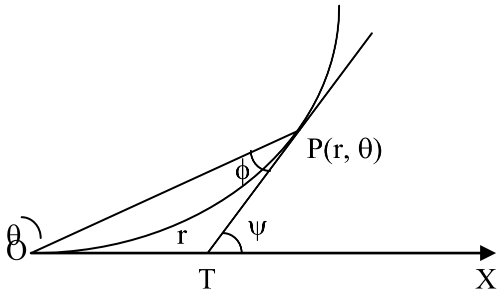
7.9 DERIVATION OF THE INTRINSIC EQUATION OF A CURVE FROM THE POLAR EQUATION

Let the polar equation of the curve be $r = f(\theta)$ …(1)

Let the curve passes through O (pole), which we take as one fixed point from where the arc OP = is measured and the initial line be the tangent to the curve at O.

If ψ = angle between the tangent at $P(r, \theta)$ and the initial line

and ϕ = angle between tangent PT and the radius vector OP, then $\psi = \theta + \phi$ …(2)



and $\tan\phi = r\dfrac{d\theta}{dr} = \dfrac{f(\theta)}{f'(\theta)}$ …(3)

$$s = \int_0^\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^\theta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta \quad \text{…(4)}$$

Eliminating θ and ϕ from (2), (3) and (4), we obtain the required intrinsic equations.

7.10. Derivation of the intrinsic equation of a curve from the pedal equation

Let the pedal equation of the curve be $p = f(r)$ …(1)

Let the curve passes through the pole O, which we take as the fixed point from where the arc OP = s is measured and the initial line be the tangent to the curve at O.

If ψ = angle between the tangent at P and the initial line, then

$$\frac{ds}{d\psi} = \rho = r\frac{dr}{dp} = \frac{r}{\frac{dp}{dr}} = \frac{r}{f'(r)} \quad \text{…(2)}$$

Also
$$s = \int_0^r \frac{r}{\sqrt{r^2 - p^2}}\, dr$$

$$= \int_0^r \frac{r}{r^2 - [f(r)]^2}\, dr \quad \text{…(3)}$$

Eliminating r between (2) and (3), we get a relation between s and $\dfrac{ds}{d\psi}$, such that

$$\frac{ds}{d\psi} = g(s)$$

or
$$\frac{d\psi}{ds} = \frac{1}{g(s)}$$

or
$$d\psi = \frac{ds}{g(s)}$$

Integrating,
$$\psi = \int \frac{ds}{g(s)} + c$$

which is a relation between s and ψ and we obtain the required intrinsic equation.

Example 1. Find the intrinsic equation of the semi-cubical parabola $ay^2 = x^3$, taking the cusp as the fixed point.

Remarks

Solution. The solution of the semi-cubical parabola is $ay^2 = x^3$ …(1)

$\therefore \quad 2ay.\dfrac{dy}{dx} = 3x^2 \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{3x^2}{2ay}$

Now

$$s = \int_0^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^x \sqrt{1+\frac{9x^4}{4a^2y^2}}dx$$

$$= \int_0^x \sqrt{1+\frac{9x^4}{4ax^3}}dx = \int_0^x \sqrt{1+\frac{9}{4a}}\,x\,dx$$

$$= \left[\frac{\left(1+\frac{9x}{4a}\right)^{3/2}}{\frac{3}{2}.\frac{9}{4a}}\right]_0^x = \frac{8a}{27}\left[\left(1+\frac{9x}{4a}\right)^{3/2}-1\right] \quad \text{…(2)}$$

Also

$$\tan\psi = \frac{dy}{dx} = \frac{3x^2}{2ay}$$

$$\therefore \quad \tan^2\psi = \frac{9x^4}{4a^2y^2} = \frac{9x^4}{4ax^3} = \frac{9x}{4a} \quad \text{…(3)}$$

Eliminating x from (2) and (3), we get

$$s = \frac{8a}{27}[(1+\tan^2\psi)^{3/2}-1]$$

or

$$= \frac{8a}{27}[\sec^3\psi - 1]$$

or $27s = 8a(\sec^3\psi - 1)$

which is the required intrinsic equations.

Example 2. Find the intrinsic equation of the cycloid $x = a(t + \sin t)$, $y = a(1 - \cos t)$ and prove that $s^2 + \rho^2 = 16a^2$.

Solution. Here $x = a(t + \sin t)$, $y = a(1 - \cos t)$

$$\frac{dx}{dt} = (1+\cos t), \quad \frac{dy}{dt} = a\sin t$$

$$\therefore \quad \frac{dy}{dx} = \frac{a\sin t}{a(1+\cos t)} = \frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{2\cos^2\frac{t}{2}} = \tan\frac{t}{2}$$

$$\therefore \quad \tan\psi = \tan\frac{t}{2} \quad \Rightarrow \quad t = 2\psi \quad \text{…(1)}$$

For $t = 0$, $x = 0$ and $y = 0$

Thus the curve passes through the origin and x-axis is the tangent at the origin

$$\therefore \quad s = \int_0^t \sqrt{a^2(1+\cos t)^2 + a^2\sin^2 t}dt$$

Remarks

$$= a\int_0^t \sqrt{2+2\cos t}\, dt = 2a\int_0^t \cos\frac{t}{2}\, dt$$

$$= 2a\left|\frac{\sin\frac{t}{2}}{\frac{1}{2}}\right|_0^t = 4a\left(\sin\frac{t}{2}\right)$$

$\therefore$ $\quad s = 4a \sin\psi$ $\quad$ [$\because$ From (1), $t = 2\psi$]

which is the required intrinsic equation.

Again $\quad \dfrac{ds}{d\psi} = 4a\cos\psi$ i.e., $\rho = 4a\cos\psi$

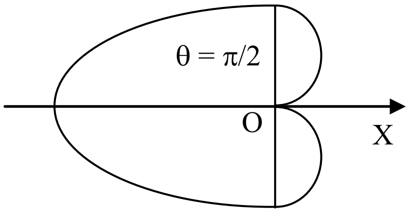
$\therefore$ $\quad s^2 + \rho^2 = 16a^2\sin^2\psi + 16a^2\cos^2\psi = 16a^2$.

Example 3. Find the intrinsic equation of the cardioid $r = a(1 - \cos\theta)$.

Solution. The equation of the cardioid is

$$r = a(1 - \cos\theta)$$

$$\therefore \quad \frac{dr}{d\theta} = a\sin\theta$$



Now $\quad s = \displaystyle\int_0^\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

$$= \int_0^\theta \sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}\, d\theta$$

$$= \int_0^\theta \sqrt{2a^2(1-\cos\theta)}\, d\theta = \int_0^\theta \sqrt{2a^2.2\sin^2\frac{\theta}{2}}\, d\theta$$

$$= \int_0^\theta 2a\sin\frac{\theta}{2}\, d\theta = -4a\left|\cos\frac{\theta}{2}\right|_0^\theta$$

$$= -4a\left[\cos\frac{\theta}{2} - 1\right] = 4a\left(1 - \cos\frac{\theta}{2}\right)$$

$$= 4a.2\sin^2\frac{\theta}{4} = 8a\sin^2\frac{\theta}{4} \qquad \ldots(1)$$

Also $\quad \tan\phi = r\dfrac{d\theta}{dr} = \dfrac{a(1-\cos\theta)}{a\sin\theta} = \tan\dfrac{\theta}{2}$

$$\therefore \quad \phi = \frac{\theta}{2} \qquad \ldots(2)$$

But $\quad \psi = \theta + \phi$

Remarks

$\therefore \quad \psi = \theta + \frac{\theta}{2} = \frac{3\theta}{2}$

Or $\quad \frac{\theta}{2} = \frac{\psi}{3}$ …(3)

Eliminating θ from (1) and (3), we get $s = 8a\sin\frac{\psi}{6}$

which is the required intrinsic equation of the curve.

Example 4. Find the intrinsic equation of the curve $p = r \sin \alpha$.

Solution. Here $\quad p = r \sin \alpha$

$$\therefore \quad \frac{dp}{dr} = \sin\alpha$$

$$\therefore \quad \frac{ds}{d\psi} = \frac{r}{\frac{dp}{dr}} = \frac{r}{\sin\alpha} \qquad \text{…(1)}$$

Also
$$s = \int_0^r \frac{r}{\sqrt{r^2 - p^2}}\,dr$$

$$= \int_0^r \frac{r}{\sqrt{r^2 - r^2\sin^2\alpha}}\,dr = \int_0^r \sec\alpha\, dr = r\sec\alpha$$

$\therefore \quad r = s \cos \alpha$

$$\therefore \text{ From (1),} \quad \frac{ds}{d\psi} = \frac{s\cos\alpha}{\sin\alpha} = s\cot\alpha$$

or
$$\frac{ds}{s} = \cot\alpha\, d\psi$$

Integrating, $\quad \log s = \psi \cot \alpha + c$ …(2)

When $\psi = 0$, let $\quad s = a$

$\therefore \quad \log a = c$

$\therefore \quad \log s = \psi \cot \alpha + \log a$ [From (2)]

or $\quad \log\frac{s}{a} = \psi \cot \alpha$

or $\quad \frac{s}{a} = e^{\psi \cot \alpha} \quad \Rightarrow \quad s = ae^{\psi \cot \alpha}$

which is the required intrinsic equation.

Remarks

Example 5. Find the intrinsic equation of the curve whose pedal equation is $p^2 = r^2 - a^3$.

Solution. The given equation of the curve is

$$p^2 = r^2 - a^3 \qquad \ldots(1)$$

$$2p\frac{dp}{dr} = 2r \quad \Rightarrow \quad p\frac{dp}{dr} = r$$

or $$\frac{p}{r} = \frac{dr}{dp}$$

$\therefore$ $$\frac{ds}{d\psi} = \rho = r\frac{dr}{dp} = r.\frac{p}{r} = p \qquad \ldots(2)$$

$\therefore$ $$s = \int_a^r \frac{r\,dr}{\sqrt{r^2 - p^2}} = \int_a^r \frac{r}{a} r \qquad \text{[Using (1)]}$$

$$= \frac{1}{2a}|\,r^2\,|_a^r = \frac{1}{2a}(r^2 - a^2) = \frac{p^2}{2a} \text{ [Using (1)]} \qquad \ldots(3)$$

or $$p^2 = 2as \quad \Rightarrow \quad p = \sqrt{2as}$$

or $$\sqrt{2as} = \frac{ds}{d\psi} \qquad \text{[Using (2)]}$$

$\therefore$ $$\frac{ds}{\sqrt{s}} = \sqrt{2a}\,d\psi$$

Integrating, $$\int s^{-1/2} ds = \sqrt{2a}\int 1 d\psi + c$$

$\Rightarrow$ $$2\sqrt{s} = \sqrt{2a}\,\psi + c \qquad \ldots(4)$$

where c is a constant of integration.

For $\psi = 0$, $s = 0$ and then from (4), $c = 0$

$$2\sqrt{s} = \sqrt{2a}\,\psi$$

or $$4s = 2a\psi^2 \qquad \text{[Squaring]}$$

or $$s = \frac{a}{2}\psi^2$$

which is the required intrinsic equation.

Exercise 7.4

1. Find the intrinsic equation of the parabola $x^2 = 4ay$

Ans. $a[\tan\psi \sec\psi + \log(\tan\psi + \sec\psi)]$.

2. Show that the intrinsic equation of the curve $r = ae^{\theta \cot \alpha}$, where s is measured from the Pt. (a, 0) is $s = a \sec \alpha(e^{\psi \cot \alpha} - 1)$.

Remarks

3. Show that the intrinsic equation of the parabola $y^2 = 4ax$ is

$$s = a \cot \psi \operatorname{cosec} \psi + a \log (\cot \psi + \operatorname{cosec} \psi).$$

4. Find the intrinsic equation of the curve $x = a(1 + \sin t)$, $y = a(1 + \cos t)$.

Ans. $s + a\psi = 0$.

5. Show that the intrinsic equation of the curve $r = ae^{m\theta}$ is $s = \dfrac{a\sqrt{1+m^2}}{m}[e^{m\psi} - 1]$.

6. Find the intrinsic equation of the parabola $x = at^2$, $y = 2at$, these being measured from the vertex.

Ans. $s = a \cot \psi \operatorname{cosec} \psi + a \log (\cot \psi + \operatorname{cosec} \psi)$.

Keywords : Arc, Cartesian, Polar, Parametric, Pedal, Intrinsic Equation.

Summary

Rectification is the process of finding the length of an arc of a curve between distinct points. If s denotes the length of the arc of a curve measured from some fixed point A to a variable point P and ψ is the angle between the tangents at P and A. Then the relation between S and ψ is called intrinsic equation of a curve. S and ψ for a point depend only upon the form of the curve and not on its position in the plane.

Remarks

CHAPTER – VIII
QUADRATURE

8.0 STRUCTURE

8.1. INTRODUCTION

We know that if on the interval [a, b], the function $f(x) \geq 0$, then the area bounded by the curve $y = f(x)$, the x-axis and the straight line $x = a$ and $x = b$ is given by the definite integral

$$A = \int_a^b f(x)\,dx$$

If $f(x) \leq 0$ on [a, b], then the definite integral $\int_a^b f(x)\,dx$ is also ≤ 0.

8.2 OBJECTIVE

After reading this chapter, you should be able to

- Understand the tracing of curves.
- Find the area bounded by curves.

8.3 Definition. The process of determining the area of a plane region is known as quadrature.

Remark. The area of the curve $x = f(y)$ between the y-axis and the lines $y = c$ to $y = d$ is given by $\int_c^d f(y)\,dy$.

Example 1. Trace the curve $x(x^2 + y^2) = a(x^2 - y^2)$ and find the area enclose by it.

Solution. The given curve is $x(x^2 + y^2) = a(x^2 - y^2)$

$$y^2(x + a) = x^2(a - x) \qquad \ldots(1)$$

Let us trace the given curve roughly.

1. The curve is symmetrical about the x-axis and there is not other symmetry.
2. The curve passes through the origin, tangents at the origin being $x^2 - y^2 = 0$ or $y = \pm x$, which area real and distinct, hence origin is a node.
3. The curve cuts the x-axis when $x^3 = ax^2$

i.e., $x^3 - ax^2 = 0$

or $x^2(x - a) = 0 \quad \Rightarrow \quad x = 0, a$

i.e., at (0, 0), (a, 0)

It crosses the y-axis only at the origin.

Shifting the origin at (a, 0); (1) transforms ito

$$Y^2(X + 2a) = (X + a)^2 (a - X - a) = -X(X + a)^2.$$

Remarks

The tangents at the new origin are given by X = 0 i.e., the new y-axis. Hence at the point (a, 0), tangent to the curve is parallel to y-axis.

4. The asymptote parallel to y-axis is given by x + a = 0 and the curve has no other asymptote.
5. Writing the equation (1) in the form

$$y^2 = \frac{x^2(a-x)}{a+x} \text{ or } y = \pm x\sqrt{\frac{a-x}{a+x}},$$

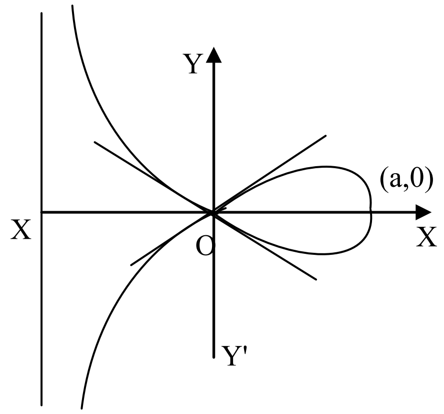
we see that y is real for $|x| \le a$ or $-a \le x \le a$ i.e., the whole of the curve lies between the lines x = –a and x = a. Thus, the shape of the curve is as shows in fig.

Now the upper half of the loop, x varies from 0 to a.

∴ Required area of the loop = 2 × arc of the upper half of the loop

$$= 2\int_0^a y\,dx$$

$$= 2\int_0^a x\sqrt{\frac{a-x}{a+x}}dx$$



Let $x = a\sin\theta$

∴ $dx = a\cos\theta\,d\theta$

When x = 0 from (2), we have θ = 0

and when x = a from (2), we have $\theta = \frac{\pi}{2}$

∴ Required area = $2\int_0^{\pi/2} a\sin\theta\sqrt{\frac{a(1-\sin\theta)}{a(1+\sin\theta)}}.a\cos\theta d\theta$

$$= 2a^2\int_0^{\pi/2}\frac{\sin\theta\cos\theta(1-\sin\theta)}{\cos\theta}d\theta$$

[Rationalising the numerator and simplifying]

$$= 2a^2\int_0^{\pi/2}(\sin\theta - \sin^2\theta)d\theta$$

$$= 2a^2\int_0^{\pi/2}\left[\sin\theta - \frac{1}{2}(1-\cos 2\theta)\right]d\theta = 2a^2\left[-\cos\theta - \frac{\theta}{2} + \frac{1}{2}.\frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= 2a^2\left[\left(-0 - \frac{\pi}{4} + 0\right) - (-1 - 0 + 0)\right] = 2a^2\left(1 - \frac{\pi}{4}\right)$$

$$= \frac{a^2}{2}(4-\pi).$$

Example 2. Find the area of the loop of the curve $y^2(a-x) = x^2(a+x)$. Also find the area of the portion bounded by the curve and its asymptote.

Solution. The given curve is $y^2(a-x) = x^2(a+x)$ …(1)

Let us make a rough sketch of the given curve.

1. The curve is symmetrical about the x-axis.
2. The curve passes through the origin, tangents at the origin being

$$y^2 = x^2 \text{ or } y = \pm x$$

which are real and different and so origin is a node.

Remarks

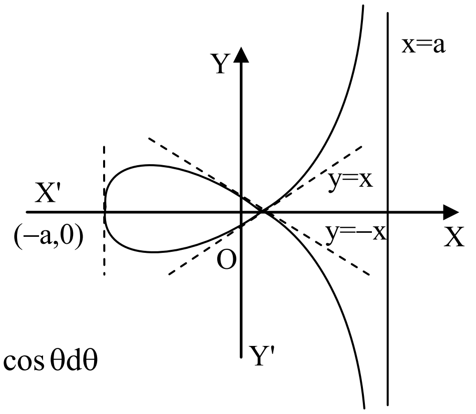
3. The curve meets x-axis where $x^2(a + x) = 0$ or $x = -a$, $x = 0$
i.e., at the points (0, 0), (–a, 0)
The curve meets y-axis at the origin .
4. The asymptote parallel to y-axis is given by $(a - x) = 0$ or $x = a$ and the curve has no other asymptote.
5. From (1), $y^2 = x^2\left(\frac{a+x}{a-x}\right)$ …(2)

L.H.S. of (2) is always positive. Thus its R.H.S. must also be positive. If $x > a$ or $x < -a$ then y^2 becomes negative. So no portion of curve lies beyond $x = \pm a$. The shape of the curve is as shown in fig.

$\therefore$ Required area = 2 × area of the loop above the x-axis

$$= 2\int_{-a}^{0} y\,dx$$

$$= 2\int_{-a}^{0} x\sqrt{\frac{a+x}{a-x}}dx$$



Put $x = a\sin\theta$; $\therefore$ $dx = a\cos\theta\, d\theta$
Now when $x = 0$, $\theta = 0$ and
when $x = -a$, $\theta = -\frac{\pi}{2}$

$\therefore$ Required area $= 2\int_{-\pi/2}^{0} a\sin\theta\sqrt{\frac{a(1+\sin\theta)}{a(1-\sin\theta)}}.a\cos\theta d\theta$

$$= 2a^2\int_{-\pi/2}^{0} \sin\theta\cos\theta\sqrt{\frac{1+\sin\theta}{1+\sin\theta}}.\sqrt{\frac{1+\sin\theta}{1+\sin\theta}}\,d\theta$$

$$= 2a^2\int_{-\pi/2}^{0} \frac{\sin\theta\cos\theta(1+\sin\theta)}{\cos\theta}d\theta$$

$$= 2a^2\int_{-\pi/2}^{0}\left[\sin\theta+\frac{1}{2}(1-\cos 2\theta)\right]d\theta$$

$$= 2a^2\left|-\cos\theta+\frac{1}{2}\theta-\frac{\sin 2\theta}{4}\right|_{-\pi/2}^{0} \quad \text{…(3)}$$

$$= 2a^2\left[(-\cos 0+0-0)-\left(-\cos\left(-\frac{\pi}{2}\right)+\frac{1}{2}\left(-\frac{\pi}{2}\right)-\frac{\sin(-\pi)}{4}\right)\right]$$

$$= 2a^2\left[-1+\frac{\pi}{4}\right]=\frac{a^2}{2}(\pi-4)$$

$$= \frac{a^2}{2}(4-\pi) \text{ numerically} \quad [4 > \pi]$$

Also for half of the area bounded by the curve and its asymptote, x varies from 0 to a.

$\therefore$ Required area between the curve and its asymptote

$$= 2\int_{0}^{a} y\,dx$$

Remarks

$$= 2\int_0^a x\sqrt{\frac{a+x}{a-x}}dx$$

$$= 2a^2\left|-\cos\theta+\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right|_0^{\pi/2} \qquad \text{[Using (3)]}$$

$$= 2a^2\left[\left(-\cos\frac{\pi}{2}+\frac{\pi}{4}-\frac{\sin\pi}{4}\right)-(\cos 0+0-\sin 0)\right]$$

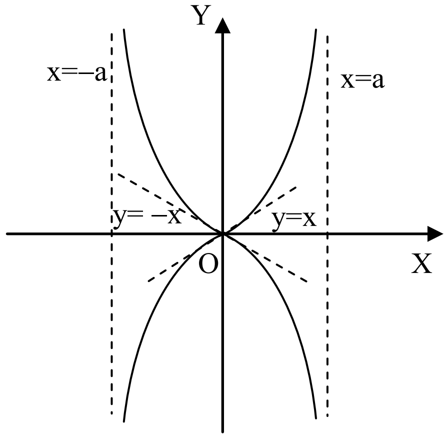
$$= 2a^2\left[\frac{\pi}{4}+1\right] = 2a^2\,\frac{(\pi+4)}{4} = \frac{a^2}{2}(\pi+4)\ .$$

Example 3. Find the area between the curve $x^2y^2 = a^2(y^2 - x^2)$ and its asymptote.

Solution. The given curve is $x^2y^2 = a^2(y^2 - x^2)$ …(1)

Let us trace the given curve roughly.

1. The curve is symmetrical about both the axes.
2. The curve passes through the origin and the tangents at the origin are given by $y^2 - x^2 = 0$ or $y = \pm x$, which are real and distinct and so the origin is a node.
3. The curve meets both x-axis and y-axis at the origin
4. The asymptotes parallel to y-axis are given by $a^2 - x^2 = 0$ and $x = \pm a$ and the curve has no other asymptote.
5. From (1), $y^2 = \dfrac{a^2x^2}{(a^2-x^2)} = \dfrac{a^2x^2}{(a+x(a-x)}$ ...(2)



L.H.S. of (2) is always positive, thus its R.H.S. must also be positive. If $x < -a$ or $x > a$, then y^2 becomes negative. Hence no portion of the curve lies beyond the lines $x = \pm a$.

Also as x increases from 0 to a, y increases from 0 to ∞. For the portion of the curve in the 1st quadrant, x varies from 0 to a.

As the curve is symmetrical about both the axes.

∴ Required area between the curve and its asymptotes

= 4[Area of the curve in the first quadrant and its asymptote]

$$= 4\int_0^a y\,dx$$

$$= 4\int_0^a \frac{ax}{\sqrt{a^2-x^2}}dx \qquad \text{[Using (2)]}$$

Put $x = a\sin\theta$ ∴ $dx = a\cos\theta\, d\theta$

Now when $x = 0$, $\theta = 0$ and when $x = a$, $\theta = \dfrac{\pi}{2}$

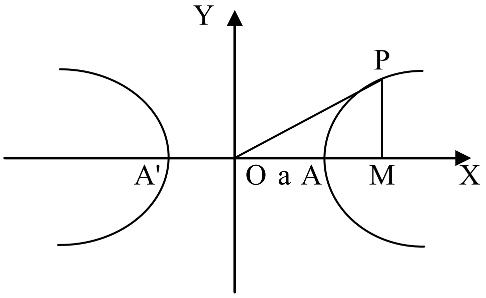
∴ Required area $= 4\displaystyle\int_0^{\pi/2}\frac{a.a\sin\theta}{a\cos\theta}.a\cos\theta d\theta$

Remarks

$$= 4a^2 \int_0^{\pi/2} \sin\theta d\theta = 4a^2 \left| -\cos\theta \right|_0^{\pi/2}$$
$$= 4a^2(0 + 1) = 4a^2.$$

Example 4. If A is the vertex, O the centre and P(x, y) and point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, prove that $x = a \cosh \frac{2S}{ab}$, $y = b \sinh \frac{2S}{ab}$, where S is the sectorial area OPA.

Solution. The given equation is



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

or

$$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1$$

Taking square root (+ve sign), we get

$$y = \frac{b}{a}\sqrt{x^2 - a^2} \qquad \ldots(1)$$

Now S = sectorial area OPA
= area OMP – area AMP ...(2)

But area OMP = $\frac{1}{2}$OM.MP

$$= \frac{1}{2}.x.y$$
$$= \frac{1}{2}x.\frac{b}{a}\sqrt{x^2 - a^2} \qquad \ldots(3)$$

Also area AMP = $\int_a^x y dx = \int_a^x \frac{b}{a}\sqrt{x^2 - a^2}dx$ [OA = a]

$$= \frac{b}{a}\left[\frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}\right]_a^x \quad \left[\int \sqrt{x^2 - a^2}dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}\right]$$
$$= \frac{b}{a}\left[\left(\frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}\right) - \left(0 - \frac{a^2}{2}\cosh^{-1}1\right)\right]$$
$$= \frac{b}{a}\left[\frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}\right] \qquad \ldots(4)$$

$\therefore$ From (2), (3) and (4), we get

$$S = \frac{1}{2}.\frac{b}{a}x\sqrt{x^2 - a^2} - \frac{b}{a}\left[\frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}\right]$$
$$= \frac{ab}{2}\cosh^{-1}\frac{x}{a}$$

Remarks

$$\therefore \quad \cosh^{-1}\frac{x}{a} = \frac{2S}{ab} \quad \Rightarrow \quad x = a\cosh\frac{2S}{ab}$$

Then from (1), $y = \frac{b}{a}\sqrt{x^2 - a^2}$

$$= \frac{b}{a}\sqrt{a^2\left(\cosh^2\frac{2S}{ab} - 1\right)}$$

$$= \frac{b}{a}.a\sinh\frac{2S}{ab} = b\sinh\frac{2S}{ab}.$$

Exercise 8.1

1. Find the area bounded by $y^2 = 4ax$ and its latus retcum.

Ans. $\frac{8a^2}{3}$.

2. Find the area between the curve $y^2 = \frac{x^3}{2a - x}$ and its asymptote.

Ans. $3\pi a^2$.

3. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Ans. πab.

4. In the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, prove that x = a cos $\frac{2S}{ab}$, $y = b\sin\frac{2S}{ab}$, where S is the sectorial area bounded by the ellipse, x-axis and the line joining (0, 0) to (x, y).

5. Find the whole area of the curve $y^2 = x^2\left(\frac{a^2 - x^2}{a^2 + x^2}\right)$.

Ans. $a^2(\pi - 2)$.

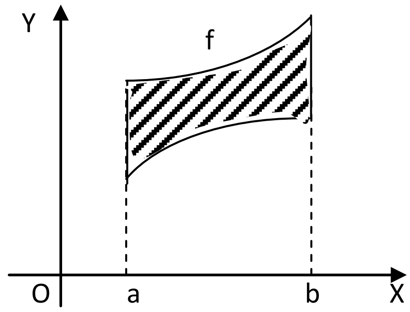
8.4 AREA BETWEEN TWO CURVES

Consider two functions f and g which are both continuous on the interval [a, b]. If f and g are both positive and if f(x) ≥ g(x) for all x in [a, b], then the area between the graphs is given by

$$A = \int_a^b f(x)dx - \int_a^b g(x)dx$$

$$= \int_a^b [f(x) - g(x)]dx$$

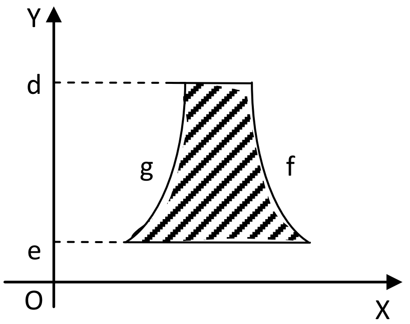
$$= \int_a^b (y_U - y_L)dx$$



where $y_I = f(x)$ is y for the upper curve and $y_{II} = g(x)$ is y for the lower curve.

Remarks. If the boundary curves are not functions of x but functions of y, then the required area is given by the integral

$$\int_c^d [f(y)-g(y)]dy$$



Remarks

Note. While using the formula obtained in above article, the geometry of the problem must be observed very carefully. As in example 2 which follow, we use $\int (y_1+y_2)dx$

Example 1. Find the area common to the parabola $y^2 = 4ax$ and $x^2 = 4ay$.

Solution. The equations of the given parabolas are

$$y^2 = 4ax \qquad \ldots(1)$$

and

$$x^2 = 4ay \qquad \ldots(2)$$

The vertices of these parabolas are at the origin and their axes are x-axis and y-axis respectively as shown fig.

Solving (1) and (2) for x and y, we get

$$\frac{x^4}{16a^2} = 4ax$$

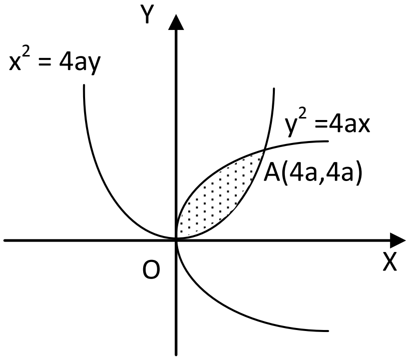
$$\Rightarrow \quad x(x^3 - 64a^3) = 0$$

$$\Rightarrow \quad x = 0, \; x = 4a$$

When x = 0, y = 0

and when x = 4a, y = 4a

$\therefore$ Thus the points of intersection of two parabola are (0, 0) and (4a, 4a).



$\therefore$ Required area (common to both the parabola)

$$= \int_0^{4a} [y_{upper} - y_{lower}]dx$$

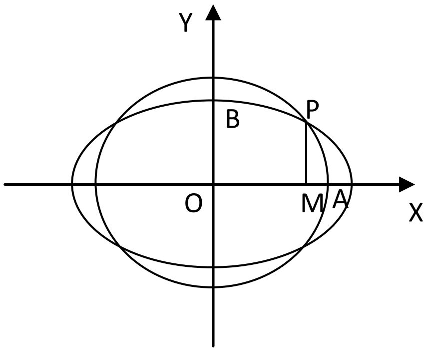
$$= \int_0^{4a} \left(\sqrt{4ax} - \frac{x^2}{4a}\right)dx \qquad \text{[From (1) and (2)]}$$

$$= \left[\frac{4\sqrt{a}}{3}x^{3/2} - \frac{1}{12a}x^3\right]_0^{4a}$$

$$= \left[\frac{4\sqrt{a}}{3}(4a)^{3/2} - \frac{1}{12a}(4a)^3\right] - 0$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3}.$$

Example 2. Find the area common to the circle $x^2 + y^2 = 4$ and the ellipse $x^2 + 4y^2 = 9$.



Solution. The given curves are

$$x^2 + y^2 = 4 \qquad \ldots(1)$$

and

$$x^2 + 4y^2 = 0 \qquad \ldots(2)$$

From (1) and (2), we get

$$x^2 + 4(4 - x^2) = 9$$

$$\Rightarrow \quad 3x^2 = 7$$

Remarks

$$\Rightarrow \qquad x = \pm\sqrt{\frac{7}{3}}$$

Take $x = \sqrt{\frac{7}{3}}$ as the point P of intersection, which we are considering lies in the 1st quadrant. Now from (1), $y = \sqrt{4-x^2} = y_I$ (say)

and from (2), $y = \frac{1}{2}\sqrt{9-x^2} = y_{II}$ (say)

Since the ellipse and the circle are symmetrical about both the axes, so required common area

$$= 4 \times \text{area OAPB} = 4[\text{area OBPM} + \text{area MPA}]$$

$$= 4\left[\int_0^{\sqrt{7/3}} y_{II}dx + \int_{\sqrt{7/3}}^{2} y_I dx\right]$$

$$= \left[\int_0^{\sqrt{7/3}} \frac{1}{2}\sqrt{9-x^2}\,dx + \int_{\sqrt{7/3}}^{2} \sqrt{4-x^2}\,dx\right]$$

$$= 2\left[\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3}\right]_0^{\sqrt{7/3}} + 4\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_{\sqrt{7/3}}^{2}$$

$$= \left[\sqrt{\frac{7}{3}}.\sqrt{9-\frac{7}{3}} + 9\sin^{-1}\sqrt{\frac{7}{27}} - 0\right] + 2\left[0 + 4\sin^{-1}1 - \sqrt{\frac{7}{3}}\sqrt{4-\frac{7}{3}} - 4.\sin^{-1}\sqrt{\frac{7}{12}}\right]$$

$$= \sqrt{\frac{7}{3}}\left[\sqrt{\frac{20}{3}} - \sqrt{\frac{20}{3}}\right] + 9\sin^{-1}\sqrt{\frac{7}{27}} + \frac{8\pi}{2} - 8\sin^{-1}\sqrt{\frac{7}{12}}$$

$$= 4\pi + 9\sin^{-1}\sqrt{\frac{7}{27}} - 8\sin^{-1}\sqrt{\frac{7}{27}}.$$

Example 3. Find the area common to the parabola $y^2 = a$ and the circle $x^2 + y^2 = 4ax$.

Solution. The given curves are $y^2 = ax$

and $\qquad x^2 + y^2 = 4ax$

Here equation (1) represents a parabola with origin as vertex, axis as x-axis and latus rectum a. Also (2) represents a circle with centre (2a, 0) and radius 2a.

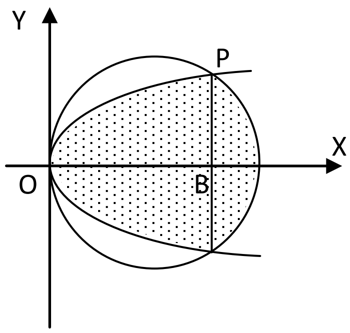
From (1) and (2), we get

$\qquad x^2 + ax = 4ax$

or $\qquad x^2 - 3ax = 0$

$\Rightarrow \qquad x(x - 3a) = 0$

$\Rightarrow \qquad x = 0, 3a$



The circle meets x-axis where

$\qquad x^2 - 4ax = 0$

i.e., $\qquad x(x - 4a) = 0$

or $\qquad x = 0$ and $x = 4a$

Now $\qquad$ y for parabola $= y_I = \sqrt{ax}$

$\qquad$ y for circle $= y_{II} = \sqrt{4ax - x^2}$

$\therefore$ Required area (common to parabola and circle)

$$= 2 \times \text{area AOP} = 2[\text{area OPB} + \text{area BPA}]$$

$$= 2\left[\int_0^{3a} y_I dx + \int_{3a}^{4a} y_{II} dx\right]$$

$$= 2\sqrt{a}\left[\int_0^{3a} \sqrt{x}dx + 2\int_{3a}^{4a} \sqrt{4a^2(x-2a)^2}dx\right]$$

$$= 2\sqrt{a}\,\frac{2}{3}[x^{3/2}]_0^{3a} + 2\left[\frac{1}{2}(x-2a)\sqrt{4a^2-(x-2a)^2} + \frac{4a^2}{2}\sin^{-1}\frac{x-2a}{2a}\right]_{3a}^{4a}$$

$$= \frac{4}{3}\sqrt{a}(3a)^{3/2} + \left[(x-2a)\sqrt{4ax-x^2} + 4a^2\sin^{-1}\frac{x-2a}{2a}\right]_{3a}^{4a}$$

$$= 4\sqrt{3}a^2 + 4a^2\sin^{-1}1 - a\sqrt{3a^2} - 4a^2\sin^{-1}\frac{1}{2}$$

$$= 4\sqrt{3}a^2 + 4a^2.\frac{\pi}{2} - \sqrt{3}a^2 - 4a^2.\frac{\pi}{6}$$

$$= 3\sqrt{3}a^2 + 2a^2\pi - \frac{2a^2\pi}{3} = 3\sqrt{3}a^2 + \frac{4\pi a^2}{3}$$

$$= a^2\left(3\sqrt{3} + \frac{4\pi}{3}\right).$$

Exercise 8.2

1. Find the area included between the parabola x^2 4ay and the curve $y(x^2 + 4a^2) = 8a^2$.
2. Find the area included between the curves $y^2 = 4bx$ and $x^2 = 4ay$.

Ans. $\frac{16ab}{3}$.

3. Show that the larger of the two areas into which the circle $x^2 + y^2 = 64a^2$ is divided by the parabola $y^2 = 12ax$ is $\frac{16}{3}a^2(8\pi - \sqrt{3})$.
4. (i) Find the area bounded by the parabola $x^2 = 8y$ and the line $x - 2y + 8 = 0$.
   (ii) Find the area bounded by the parabola $y = 2 - x^2$ and the straight line $y = -x$.

Ans. (i) 36 (numerically) (ii) $\frac{17}{2}$.

5. Show that the larger of the two area into which the circle $x^2 + y^2 = 64a^2$ is divided by the parabola $y^2 = 12ax$ is $\frac{16}{3}a^2(8\pi - \sqrt{3})$.

8.5. Area Formula for Parametric Curves

If the functions f, g have continuous derivatives, then the area bounded by the curves $x = f(t)$, $y = g(t)$, the x-axis and the ordinates of the points where $t = a$, $t = b$ is

$$\int_a^b y\frac{dx}{dt}dt$$

We know that

Remarks

$$\text{Required area} = \int_{\alpha}^{\beta} y\,dx, \quad \text{where } x = \alpha, \text{ if } t = a \text{ and } x = \beta \text{ if } t = b$$

$$= \int_{a}^{b} y\frac{dx}{dt}dt$$

Note. The area bounded by the curves x = f(t), y = g(t), the y-axis and the abscissae at the points when t = c, t = d is $\int_{c}^{d} x\frac{dy}{dt}dt$.

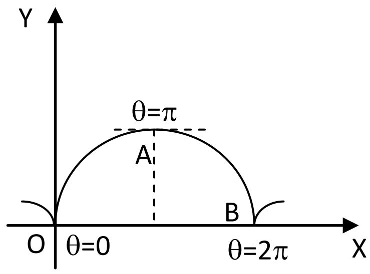
Example 1. Find the area included between the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ and its base.

Solution. For the first half of the cycloid, θ varies from 0 to π.

$$\therefore \qquad \text{Required area} = 2\int_{0}^{\pi} y.\frac{dx}{d\theta}d\theta$$

$$= 2\int_{0}^{\pi} a(1-\cos\theta).a(1-\cos\theta)d\theta$$

$$\left[y = a(1 - \cos\theta) \text{ and } \frac{dx}{d\theta} = a(1 - \cos\theta)\right]$$



$$= 2a^2\int_{0}^{\pi}(1-\cos\theta)^2 d\theta$$

$$= 2a^2\int_{0}^{\pi}\left(2\sin^2\frac{\theta}{2}\right)^2 d\theta$$

$$= 2a^2\int_{0}^{\pi}4\sin^4\frac{\theta}{2}d\theta$$

Put $\frac{\theta}{2} = t$ i.e., $\theta = 2t$ so that $d\theta = 2dt$

Now when $\theta = 0$, $t = 0$ and when $\theta = \pi$, $t = \frac{\pi}{2}$

$$\therefore \qquad \text{Reqd. area} = 8a^2\int_{0}^{\pi/2}\sin^4 t.2dt = 16a^2\int_{0}^{\pi/2}\sin^4 t\,dt$$

$$= 16a^2.\frac{3.1}{4.2}.\frac{\pi}{2} = 3\pi a^2 .$$

Example 2. Find the area of the curve $x = a\cos^3 t$, $y = b\sin^3 t$

The curve is symmetrical about both the axis.

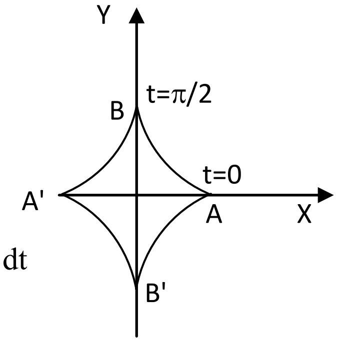
Also $\quad x = a\cos^3 t \quad \Rightarrow \quad |x| \le a$

and $\quad y = b\sin^3 t \quad \Rightarrow \quad |y| \le b$

So the curve lies within the rectangle bounded by $x = \pm a$ and $y = \pm b$.

Remarks

Also in the first quadrant t varies from 0 to $\frac{\pi}{2}$.



∴ Required area = 4 × [area OAB]

$$= 4\int_0^{\pi/2} y\left(\frac{dx}{dt}\right)dt$$

$$= 4\int_0^{\pi/2} b\sin^3 t 3a\cos^2 t(-\sin t)\,dt$$

$$= -12ab\int_0^{\pi/2} \sin^4 t\cos^2 t\,dt$$

$$= -12ab.\frac{3.1.1}{6.4.2}.\frac{\pi}{2} = \frac{3}{8}\pi ab \text{ (numerically).}$$

Exercise 8.3

1. Find the area included between the cycloid x = a(θ + sin θ), y = a(1 – cos θ) and its base.
Ans. $3\pi a^2$.
2. Find the area of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ or $x = a\cos^3 t$, $y = a\sin^3 t$.
Ans. $\frac{3}{8}\pi a^2$.
3. Find the area of the loop of the curve $x = a(1 - t^2)$, $y = at(1 - t^2)$.
Ans. $\frac{8}{15}a^2$.
4. Show that the area of the ellipse x = a cos t, y = b sin t is πab.
5. Find the whole area of the curve $x = a\left(\frac{1-t^2}{1+t^2}\right)$, $y = \frac{2at}{1+t^2}$.
Ans. πa^2.
6. Show that the area bounded by the cissoid $x = a\sin^2 t$, $y = a\frac{\sin^3 t}{\cos t}$ and its asymptote is $\frac{3\pi a^2}{4}$.

8.6. AREA FORMULA FOR POLAR CURVES (SECTORIAL AREA)

If r = f(θ) is the equation of a curve in polar co–ordinates, then the area of the section enclosed by the curve and the two radii vectors θ = α and θ = β is

$$\frac{1}{2}\int_\alpha^\beta r^2\,d\theta$$

Remark. While doing the problems for polar curves, the following points should be carefully remembered.

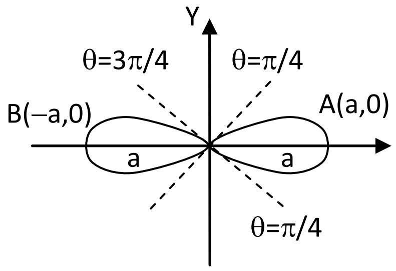
1. Sometimes it is easy to transform a Cartesian equation to polar form than to solve for y.
2. To get the limits of integration for a loop, find two successive values of θ for which r = 0.
3. If the curve is symmetrical about the initial line (i.e., x-axis) only, evaluate the integral from 0 to π and multiply the result by 2.

Remarks

4. If the curve is symmetrical about both the axes, evaluate the integral from 0 to $\frac{\pi}{2}$ and multiply the result by 4.

5. The curve r = a cos nθ or r = a sin nθ have n equal loops if n is odd and 2n equal loops if n is even.

Example 1. Find the area of a loop of the curve $r^2 = a^2 \cos 2\theta$ and hence find its total area.



Solution. The given curve is $r^2 = a^2 \cos 2\theta$ (1)

From (1), we find that on changing θ to –θ or π–θ, the equation remains unchanged. So that curve is symmetrical about both the axes.

For the loop, put r = 0 so that cos 2θ = 0

$$\therefore \qquad 2\theta = -\frac{\pi}{2}, \frac{\pi}{2}$$

$$\text{or} \qquad \theta = -\frac{\pi}{4}, \frac{\pi}{4}$$

So a loop lies between the radii vectors. The curve has two equal loops and in the first quadrant, for half the loop, θ varies from 0 to $\frac{\pi}{4}$

$$\therefore \qquad \text{Area of a loop} = 2\int_0^{\pi/4} \frac{1}{2} r^2 d\theta$$

$$= \int_0^{\pi/4} a^2 \cos 2\theta d\theta$$

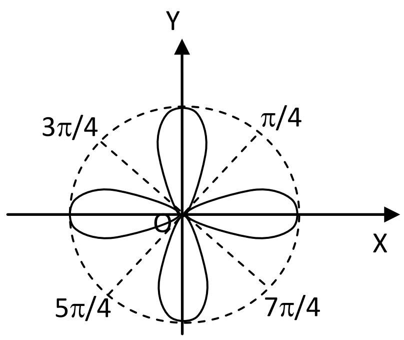
$$= a^2 \left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4} = \frac{a^2}{2}(1-0) = \frac{a^2}{2}$$

$$\therefore \qquad \text{Total area of the curve} = 2 \times \frac{a^2}{2} = a^2.$$

Example 2. Find the area of a loop of the curve r = a cos 2θ and hence find the total area of the curve.

Solution. The curve r = a cos 2θ has four equal loops.

For a lop, putting r = 0, we get cos 2θ = 0



$$\therefore \qquad 2\theta = \frac{-\pi}{2}, \frac{\pi}{2}$$

$$\text{or} \qquad \theta = \frac{-\pi}{4}, \frac{\pi}{4}$$

i.e., for the first loop of the curve,

θ varies from $\frac{-\pi}{4}$ to $\frac{\pi}{4}$

Remarks

$\therefore$ Area of one loop of the curve = $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{1}{2} r^2 d\theta$

$$= \frac{1}{2}\int_{-\pi/4}^{\pi/4} a^2 \cos^2 2\theta d\theta$$

$$= \frac{a^2}{2} \times 2\int_{0}^{\pi/4} \cos^2 2\theta d\theta$$

$$\left[\because \int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)\,dx \text{ if } f(x) \text{ is an even function of } x\right]$$

$$= a^2 \int_0^{\pi/2} \cos^2 t \frac{dt}{2} \qquad \text{[Putting } 2\theta = t\text{]}$$

$$\frac{a^2}{2}.\frac{1}{2}.\frac{\pi}{2} = \frac{\pi a^2}{8}$$

$\therefore$ Total area of the curve = 4 × (Area of one loop)

$$= 4 \times \frac{\pi a^2}{8} = \frac{\pi a^2}{2} .$$

Example 3. Find the area of the loop of the folium of Descates, $x^3 + y^3 = 3axy$.

Solution. The given equation of the curve is $x^3 + y^3 = 3axy$ …(1)

Let us put $x = r \cos \theta$, $y = r \sin \theta$, so that (1) reduces to polar form as

$$r^3(\sin^3\theta + \cos^3\theta) = 3ar^2 \sin \theta \cos \theta$$

or $$r = \frac{3a \sin\theta\cos\theta}{(\sin^3\theta + \cos^3\theta)} \qquad \text{…(2)}$$

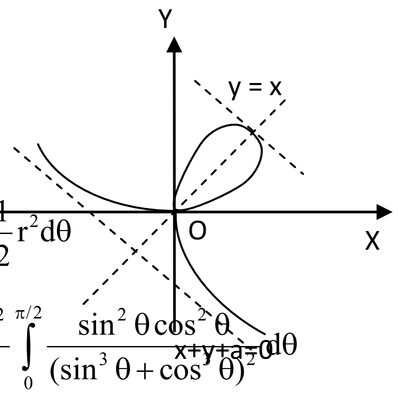
For the loop, put $r = 0$, so that $\sin \theta \cos \theta = 0$

$\Rightarrow$ $\sin \theta = 0$ i.e., $\theta = 0$

$\Rightarrow$ $\cos \theta = 0$ i.e., $\theta = \dfrac{\pi}{2}$

So for the loop, θ varies from 0 to $\dfrac{\pi}{2}$

$\therefore$ Required area of the loop



$$= \int_0^{\pi/2} \frac{1}{2} r^2 d\theta$$

$$= \frac{9a^2}{2}\int_0^{\pi/2} \frac{\sin^2\theta\cos^2\theta}{(\sin^3\theta + \cos^3\theta)^2} d\theta$$

Remarks

$$= \frac{9a^2}{2}\int_0^{\pi/2} \frac{\tan^2\theta \sec^2\theta}{(1+\tan^3\theta)^2}d\theta$$

Put $\tan^3\theta = t$, so that $3\tan^2\theta.\sec^2\theta\, d\theta = dt$

$\therefore$ Required area = $\frac{3a^2}{2}\int_0^{\infty}\frac{dt}{(1+t^2)} = \frac{3a^2}{2}\left[-\frac{1}{(1+t)}\right]_0^{\infty}$

$$= -\frac{3a^2}{2}[0-1] = \frac{3a^2}{2}.$$

Exercise 8.4

1. Find the area of one loop of the curve $r = a \sin 3\theta$.

Ans. $\frac{\pi a^2}{4}$.

2. Show that the area of a loop of the curve $r = \sqrt{3}\cos 3\theta + \sin 3\theta$ is $\frac{\pi}{3}$.

Ans. $\frac{\pi a^2}{16}\left(\frac{\pi^2}{6}-1\right)$.

3. Find the area of one loop and hence the total area for the following curves :

   (i) $r = a \sin 2\theta$ (ii) $r = a \sin 4\theta$

Ans. (i) $\frac{\pi a^2}{8};\frac{\pi a^2}{2}$ (ii) $\frac{\pi a^2}{16};\frac{\pi a^2}{2}$

4. Find the area bounded by the following cardioids :

   (i) $r = a(1+\cos\theta)$ (ii) $r = a(1-\cos\theta)$

5. Find the area of a loop of curve $r = a\theta\cos\theta$ between $\theta = 0$ and $\theta = \frac{\pi}{2}$.

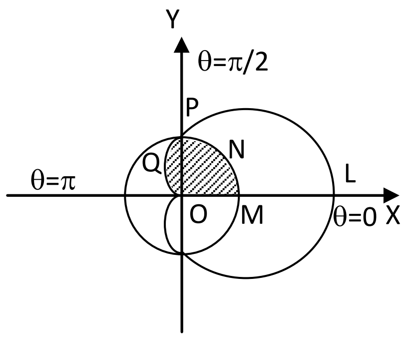
8.7. AREA BETWEEN TWO POLAR CURVES

To prove that the area bounded by the curves $r = f(\theta)$, $r = g(\theta)$ and the radii vectors $\theta = \alpha$, $\theta = \beta$

$$\int_{\alpha}^{\beta}\frac{1}{2}(r_1^2 - r_2^2)d\theta$$

where r_1 and r_2 are the 'r' for the outer and inner curves respectively.

Example 1. Find the area common to the circle $r = a$ and the cardioid $r = a(1+\cos\theta)$.



Solution. The equation of the given curves are

$$r = a \quad \ldots(1)$$

and $$r = a(1+\cos\theta) \quad \ldots(2)$$

Solving (1) and (2) to get the points of intersection, we have

$$a = a(1+\cos\theta)$$

$\Rightarrow$ $$\cos\theta = 0$$

Remarks

$\Rightarrow \qquad \theta = \pm\frac{\pi}{2}$

So, $\theta = \pm\frac{\pi}{2}$ corresponds to the points

where the two curves intersect.

$\therefore$ Required area = 2[Area OMNPQO]

= 2[Area OMNPO + Area OPQO] …(3)

Now area OMNPO = $\int_0^{\pi/2} \frac{1}{2} r^2 d\theta$ for r = a

$$= \int_0^{\pi/2} \frac{1}{2} a^2 d\theta = \frac{a^2}{2}[\theta]_0^{\pi/2}$$

$$= \frac{a^2}{2}\left(\frac{\pi}{2}\right) = \frac{\pi a^2}{4} \qquad \text{…(4)}$$

Area OPQO = $\int_{\pi/2}^{\pi} \frac{1}{2} r^2 d\theta$ for r = a(1 + cos θ)

$$= \frac{1}{2}a^2 \int_{\pi/2}^{\pi} (1+\cos\theta)^2 d\theta$$

$$= \frac{1}{2}a^2 \int_{\pi/2}^{\pi} (1+2\cos\theta+\cos^2\theta) d\theta$$

$$= \frac{a^2}{2} \int_{\pi/2}^{\pi} \left(1+2\cos\theta+\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= \frac{a^2}{2} \int_{\pi/2}^{\pi} \left(\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta\right) d\theta$$

$$= \frac{a^2}{2}\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]_{\pi/2}^{\pi}$$

$$= \frac{a^2}{2}\left[\frac{3\pi}{2}-\left(\frac{3\pi}{4}+2\right)\right] = \frac{a^2}{8}(3\pi-8) \qquad \text{…(5)}$$

Using the values from (4) and (5) in (3), we get

$$\text{Required area} = 2\left[\frac{\pi a^2}{4}+\frac{a^2}{8}(3\pi-8)\right] = a^2\left(\frac{5\pi}{4}-2\right).$$

Remarks

Example 2. Show that area of the region included between the cardioids

r = a(1 + cos θ) and r = a (1 – cos θ) is $\frac{a^2}{2}(3\pi - 8)$.

Solution. The given equation of the cardioids are

$$r = a\,(1 + \cos\theta) \qquad \ldots(1)$$

and

$$r = a(1 - \cos\theta) \qquad \ldots(2)$$

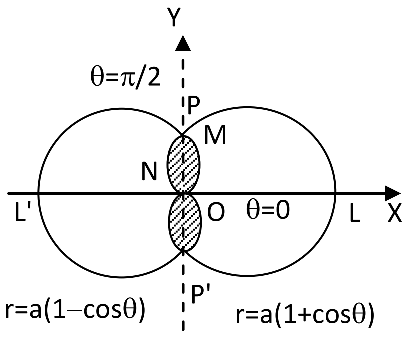
Let us solve (1) and (2) to get the points of intersection

$$a(1 + \cos\theta) = a(1 - \cos\theta)$$

$\Rightarrow$ $\quad 2\cos\theta = 0$

$\Rightarrow$ $\quad \cos\theta = 0$

$\therefore$ $\quad \theta = \pm\frac{\pi}{2}$



Also both the cardioids are symmetrical about the initial line

$\therefore$ By symmetry,

Required area common between two cardioids

$$= 2[\text{Area OMPNO}] = 4[\text{Area OMPO}]$$

where are OMPO represents the area of curve (2) between the radii vectors θ = 0 and $\theta = \frac{\pi}{2}$

$$\therefore \qquad \text{Required area} = 4\int_0^{\pi/2} \frac{1}{2} r^2 d\theta$$

$$= 2\int_0^{\pi/2} a^2 (1 - \cos\theta)^2 d\theta$$

$$= 2a^2 \int_0^{\pi/2} (1 - 2\cos\theta + \cos^2\theta) d\theta$$

$$= 2a^2[\theta - 2\sin\theta]_0^{\pi/2} + 2a^2 . \int_0^{\pi/2} \cos^2\theta d\theta$$

$$= 2a^2\left[\frac{\pi}{2} - 2\right] + 2a^2 . \frac{1}{2} . \frac{\pi}{2}$$

$$= 2a^2\left(\frac{\pi}{2} - 2 + \frac{\pi}{4}\right) = 2a^2\left(\frac{3\pi}{4} - 2\right)$$

$$= \frac{a^2}{2}(3\pi - 8).$$

Remarks

Exercise 8.5

1. Prove that the area common to the circles $r = a\sqrt{2}$ and $r = 2a\cos\theta$ is $a^2(\pi - 1)$.
Ans. $a^2(\pi - 1)$.

2. Find the area inside the circle $r = \sin\theta$ and outside the cardioid $r = (1 - \cos\theta)$.
Ans. $\left(1 - \frac{\pi}{4}\right)$.

3. Find the area outside the circle $r = 2a\cos\theta$ and inside the cardioid $r = a(1 + \cos\theta)$.
Ans. $\frac{\pi a^2}{2}$.

Keywords : Tracing, area, parametric, polar.

SUMMARY

Area is evaluated by use of definite integral

$$A = \int_a^b f(x)dx \quad \text{(bounded by curve and x-axis)}$$

$$\text{or } \int_c^d g(y)dy \quad \text{(bounded by curve and y-axis)}$$

Area bounded by two curves can be calculated by

$$A = \int_a^b (y_U - y_L)dx \qquad y_U = y_{upper}, \quad y_L = y_{Lower}$$

$$\text{or } \int_c^d (x_U - x_L)dy \qquad x_U = x_{upper}, \quad x_L = x_{Lower}$$

CHAPTER – IX

VOLUMES AND SURFACES OF SOLIDS OF REVOLUTION

9.0 STRUCTURE

9.1 INTRODUCTION

We shall discuss how the definite integral is used to find volumes of solids of revolution and area of surfaces of solids of revolution about any axis. A solid of revolution is obtained by revolving a region in a plane about a line in the plane. A surface of revolution is obtained by rotating a plane curve about a fixed line lying in its plane.

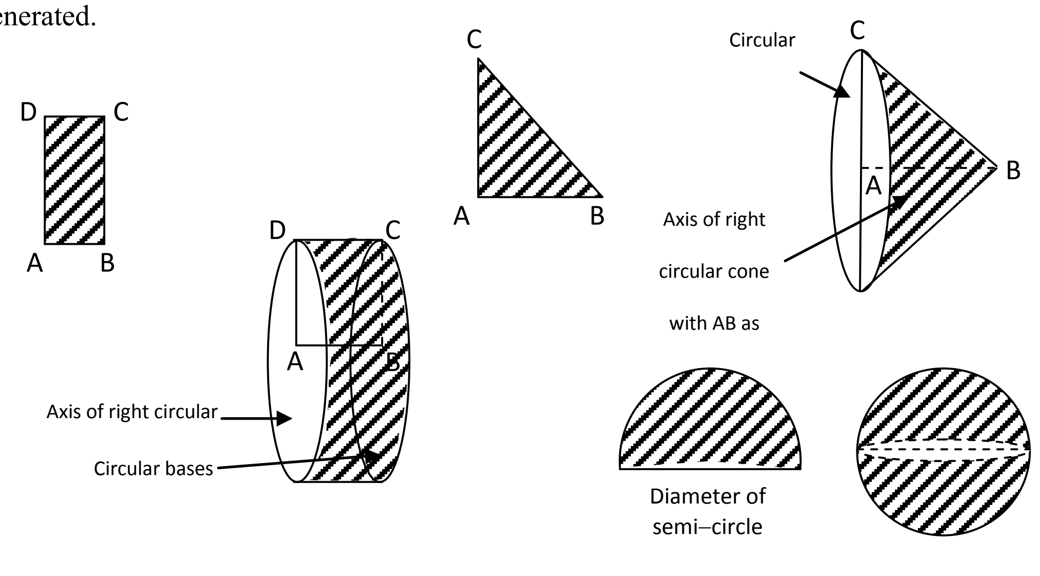
9.2 OBJECTIVE

After reading this chapter, you must be able to

- Understand, how to find volume of solids of revolution about any axis.
- Calculate the area of the surfaces of solids of revolution about any axis.

9.3 DEFINITION.A solid of revolution is obtained by revolving a region in a plane about a line in the plane called the **axis of revolution**, which either touches the boundary of the region of does not intersect the region a tall.

For example, if a rectangular region ABCD is rotated about the side AB, then a right circular cylinder is generated.



Remarks

If the region inside a right-angles triangle is rotated about one of the sides say AB, then a right circular cone is generated. Also if the region bounded by a semi-circle and its diameter is revolved about that diameter it sweeps out a sphere.

Volume of a Solid of Revolution

Volume obtained by revolving about x-axis, the arc of the curve y = f(x) intercepted between the points whose abscissae are a, b is $\int_a^b \pi y^2\,dx$ it being assumed that arc does not cut x-axis.

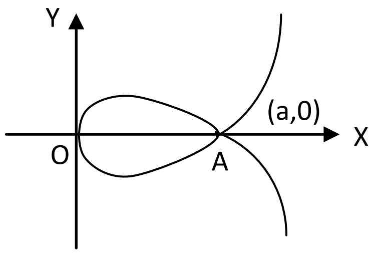
Note 1. It follows from above that the volume obtained by revolving about y-axis, the arc of a curve x = f(y) intercepted between the points whose ordinates are a, b is $\int_a^b \pi x^2\,dy$ it being assumed that the arc does not cut y-axis.

Example 1. The loop of the curve $2ay^2 = x(x-a)^2$ revolves about x-axis. Find the volume of the solid so generated.

Solution. The given equation of the curve is $2ay^2 = x(x-a)^2$ …(1)

To find the limits of integration, let us trace the curve roughly.

1. The curve is symmetrical about x-axis.
2. The curve passes through the origin and the equation of tangent at the origin is x = 0, i.e., y-axis.
3. The curve meets the x-axis in (0, 0) and (a, 0). The curve meets the y-axis at the origin only.
4. The curve has no asymptotes.
5. From (1), $y^2 = \dfrac{x(x-a)^2}{2a}$ …(2)



L.H.S. of (2) is always positive. So its R.H.S. must also be positive. It is possible if $x \ge 0$. So no portion of the curve lies to the left of the y-axis. The shape of the curve is as shows in fig. For upper half of the loop, x varies from 0 to a.

$$\therefore \quad \text{Required volume} = \int_0^a \pi y^2\,dx = \pi\int_0^a \frac{x(x-a)^2}{2a}\,dx$$

$$= \frac{\pi}{2a}\int_0^a (x^3 - 2ax^2 + a^2x)\,dx$$

$$= \frac{\pi}{2a}\left[\frac{x^4}{4} - 2a\frac{x^3}{3} + a^2\frac{x^2}{2}\right]_0^a$$

$$= \frac{\pi}{2a}\left[\frac{a^4}{4} - \frac{2a^4}{3} + \frac{a^4}{2}\right]$$

$$= \frac{\pi}{2a}.\frac{a^4}{12} = \frac{\pi a^3}{24}.$$

Example 2. The circle $x^2 + y^2 + a^2$ is revolved about the x-axis. Find the volume of the sphere so formed.

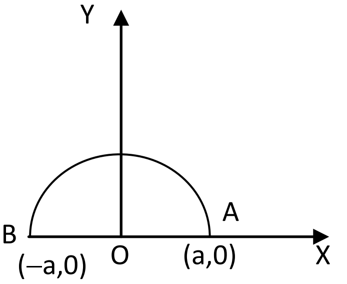
Solution. We know that the sphere is the solid of revolution generation by the revolution of a semi-circular area about its diameter.

Now, the circle of radius a is $x^2 + y^2 = a^2$ …(1)

Also for the semi-circle above the x-axis, x varies from from –a to a.

Remarks

$\therefore$ Required volume $= \int_{-a}^{a} \pi y^2 dx$



$$= \pi \int_{-a}^{a} (a^2 - x^2) dx \text{ [Using (1)]}$$

$$= \pi \left| a^2 x - \frac{x^3}{3} \right|_{-a}^{a}$$

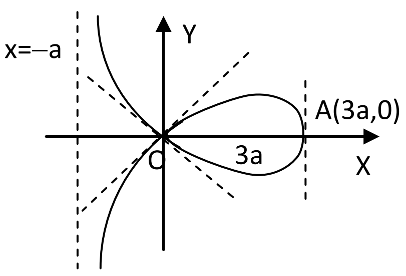
$$= \pi \left[\left(a^3 - \frac{a^3}{3} \right) - \left(-a^3 + \frac{a^3}{3} \right) \right]$$

$$= \pi \left[2a^3 - \frac{2a^3}{3} \right] = \frac{4}{3} \pi a^3.$$

Example 3. Find the volume of the solid formed by the revolution about the x-axis of the curve $y^2(a + x) = x^2(3a - x)$.

Solution. The given curve is $y^2(a + x) = x^2(3a - x)$ …(1)

1. The curve is symmetrical about the x-axis.
2. The curve passes through the origin and the tangents at the origin are $y^2 = x^2$ or $y = \pm x$, which are real and distinct and so origin is a node.
3. The curve meets x-axis in (0, 0) and (3a, 0). The curve meets y-axis at origin only.
4. The only asymptote to the curve is parallel to y-axis and is given by $a + x = 0$ i.e., $x = -a$.



5. From (1), $y^2 = \dfrac{x^2(3a - x)}{(a + x)}$ …(2)

L.H.S. of (2) is always positive. So its R.H.S. must also be positive. Now when $x > 3a$ or when $x < -a$, R.H.S. becomes negative.

Thus no portion of the curve lies beyond the lines $x = -a$ and $x = 3a$.

For upper half of the loop, x varies from 0 to 3a.

$\therefore$ Required volume $= \int_0^{3a} \pi y^2 \, dx = \pi \int_0^{3a} \dfrac{x^2(3a - x)}{x + a} dx$

$$= \pi \int_0^{3a} \frac{3ax^2 - x^3}{x + a} dx$$

$$= \pi \int_0^{3a} \left[-x^2 + 4ax - 4a^2 + \frac{4a^3}{x + a} \right] dx$$

$$= \pi \left[-\frac{x^3}{3} + 2ax^2 - 4a^2 x + 4a^3 \log(x + a) \right]_0^{3a}$$

$$= \pi[-9a^3 + 18a^3 - 12a^3 + 4a^3 \log 4a - 4a^3 \log a]$$

$$= \pi a^3 \left[4 \log \left(\frac{4a}{a} \right) - 3 \right]$$

$$= \pi a^3 [4 \log 4 - 3]$$

$$= \pi a^3 [8 \log 2 - 2].$$

Remarks

Exercise 9.1

1. Find the volume of the solid obtained by revolving x = a cos θ, y = b sin θ about the y-axis.

Ans. $\frac{4}{3}\pi a^2 b$.

2. Find the volume of the solid formed by the revolution about the x-axis of the curve $a^2 y^2 = x^2(2a - x)(x - a)$.

Ans. $\frac{23\pi a^3}{60}$.

3. Find the volume of the solid generated by revolving the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about the x-axis.

Ans. $\frac{4}{3}\pi ab^2$.

4. The area of a parabola $y^2 = 4ax$ lying between the vertex and the latus rectum is revolved about the x-axis. Find the volume generated.

Ans. $2\pi a^3$.

5. Find the volume of the solid generated by the revolution of the curve $y(a^2 + x^2) = a^3$ about its asymptote.

Ans. $\frac{\pi^2 a^3}{2}$.

9.4. ANY AXIS OF REVOLUTION

The volume of the solid generated by the resolution about any axis MN of the area bounded by the curve AB, the axis MN and the perpendiculars AM, BN on the axis

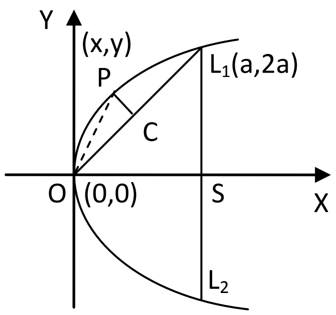
$$\int_{PM}^{ON} \pi(PC)^2 \, d(OC)$$

where O is a fixed point, the axis MN and PC is perpendicular from any point P of the curve AB on MN.

Example 1. Find the volume of the spindle formed by the revolution of a parabolic arc about the line joining the vertex to one extremity of the latus rectum.

Solution. The equation of the parabola is

$$y^2 = 4ax \qquad \ldots(1)$$



The extremities of the parabola are $L_1(a, 2a)$, $L_2(a, -2a)$ and the vertex is O(0, 0).

Now the equation of line OL_1 is

$$(y - 0) = \frac{2a-0}{a-0}(x-0)$$

or $$2x - y = 0$$

Let P(x, y) be any point on the arc OL_1 of the parabola. Draw $PL \perp OL_1$ and join OP.

∴ PC = perpendicular distance of P(x, y) from the line 2x – y = 0

$$= \frac{2x-y}{\sqrt{4+1}} = \frac{2x-2\sqrt{ax}}{\sqrt{5}}$$

$$= \frac{2\sqrt{x}(\sqrt{x}-\sqrt{a})}{\sqrt{5}}$$

Remarks

or $$PC^2 = \frac{4x(x+a-2\sqrt{ax})}{5}$$

$$\therefore \quad OC^2 = OP^2 - PC^2 = (x^2+y^2) - \frac{4x(x+a-2\sqrt{ax})}{5}$$

$$= \frac{1}{5}(x^2 + 16ax + 8\sqrt{a}\,x^{3/2}) = \frac{1}{5}(x+4\sqrt{ax})^2$$

$$OC = \frac{1}{\sqrt{5}}(x+4\sqrt{ax})$$

$$\therefore \quad d(OC) = \frac{1}{\sqrt{5}}\left(1+4\sqrt{a}.\frac{1}{2\sqrt{x}}\right) = \frac{1}{\sqrt{5}}\left(\frac{\sqrt{x}+2\sqrt{a}}{\sqrt{x}}\right)$$

Also for the arc OL_1, x varies from 0 to a.

$$\therefore \quad \text{Required volume} = \int \pi(PC)^2.d(OC)$$

$$= \pi\int_0^a \frac{4a(\sqrt{x}-\sqrt{a})^2}{5}.\frac{1}{5}\left(\frac{\sqrt{x}+2\sqrt{a}}{\sqrt{x}}\right)dx$$

$$= \frac{4\pi}{5\sqrt{5}}\int_0^a \sqrt{x}(\sqrt{x}-\sqrt{a})^2.(\sqrt{x}+2\sqrt{a})dx$$

Put $x = at^2 \therefore\ dx = 2at\,dt$

Now, when x = 0, t = 0 and when x = a, t = 1

$$\therefore \quad \text{Required volume} = \frac{4\pi}{5\sqrt{5}}\int_0^1 \sqrt{a}\,t(\sqrt{a}t-\sqrt{a})^2.(\sqrt{a}t+2\sqrt{a})2at\,dt$$

$$= \frac{8\pi a^3}{5\sqrt{5}}\int_0^1 t^2(t+2)(t^2-2t+1)dt$$

$$= \frac{8\pi a^3}{5\sqrt{5}}\int_0^1 t^2(t^3-3t+2)dt = \frac{8\pi a^3}{5\sqrt{5}}\left|\frac{t^6}{6} - 3.\frac{t^4}{4} + 2\frac{t^3}{3}\right|_0^1$$

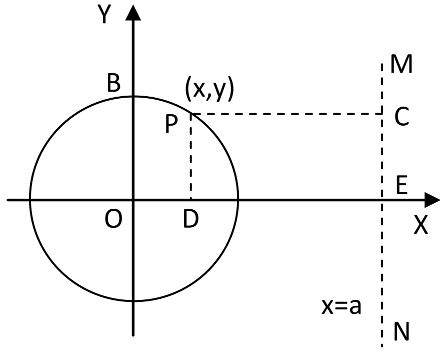
$$= \frac{8\pi a^3}{5\sqrt{5}}\left[\frac{1}{6}-\frac{3}{4}+\frac{2}{3}\right] = \frac{8\pi a^3}{5\sqrt{5}}\left[\frac{2-9+8}{12}\right]$$

$$= \frac{2\pi a^3}{15\sqrt{5}}\ .$$

Exercise 2. Find the volume of the torus obtained by rotating the area bounded by the circle $x^2 + y^2 = a^2$ around the line x = b (b > a).

Solution. The given equation of the circle is

$$x^2 + y^2 = a^2 \qquad \ldots(1)$$



Now the area enclosed by the circle revolves about the given line x = b.

Let P(x, y) be any point on the arc AB. Draw PC⊥MN and PD⊥OA.

PC = DE = OE – OD

= (b – x)

and EC = DP = y

Remarks

and for the arc AB, y varies from 0 to a

$\therefore$ Required volume = $2\int \pi(PC)^2 d(EC)$

$$= 2\int_0^a \pi(b-x)^2 dy$$

$$= 2\pi\int_0^a (b^2+x^2-2bx)dy$$

$$= 2\pi\int [(b^2+(a^2-y^2)-2b\sqrt{a^2-y^2}]dy \qquad \text{[Using (1)]}$$

$$= 2\pi\int_0^a [(b^2+a^2)-y^2-2b\sqrt{a^2-y^2}]dy$$

$$= 2\pi\left[(b^2+a^2)y-\frac{y^3}{3}-2b\left(\frac{y}{2}\sqrt{a^2-y^2}+\frac{a^2}{2}\sin^{-1}\frac{y}{a}\right)\right]_0^a$$

$$= 2\pi\left[(b^2+a^2)a-\frac{a^3}{3}-a^2b\sin^{-1}1+0\right]$$

$$= 2\pi\left[ab^2+a^3-\frac{a^3}{3}-a^2b.\frac{\pi}{2}\right]$$

$$= 2\pi\left[ab^2+\frac{2}{3}a^3-a^2b.\frac{\pi}{2}\right]=\frac{\pi}{3}(6ab^2+4a^3-3a^2b\pi].$$

Exercise 9.2

1. Find the volume of the solid generated by the revolution of the curve $(a - x)y^2 = a^2x$ and its asymptote.

Ans. $\frac{\pi^2a^3}{2}$.

2. The ellipse $b^2x^2 + a^2y^2 = a^2b^2$ is divided into two parts by the line $x = \frac{a}{2}$ and the smaller part is rotated through four right angles about this line. Prove that the volume generated is $\pi a^2b\left(\frac{3\sqrt{3}}{4}-\frac{\pi}{3}\right)$.

3. Show that the volume of the solid formed by the revolution of the cissoid $y^2(2a - x) = x^3$ about its asymptote is $2\pi^2a^3$.

4. A quadrant of a circle of radius a revolves about its chord. Show that the volume of the spindle thus generated is $\frac{\pi}{6\sqrt{2}}(10-3\pi)a^3$.

Remarks

9.5. VOLUME FORMULA FOR TWO SOLIDS

The volume of the solid generated by the revolution about the x-axis of the area bounded by the curves $y = f(x)$, $y = g(x)$ and the ordinates $x = a$, $x = b$ is

$$\int_a^b \pi(y_I^2 - y_{II}^2)\,dx$$

here y_I corresponds to y of the upper curve and y_{II} corresponds to y of the lower curve.

9.6. VOLUME FORMULA FOR PARAMETRIC CURVES

9.6.1. The volume obtained by revolving about x-axis of the region bounded by the curve $x = f(t)$, $g(t)$, the x-axis and the ordinate, where $t = a$, $t = b$ is

$$\int_a^b \pi y^2 \frac{dx}{dt}\,dt$$

9.6.2. The volume obtained by revolving about the y-axis of the region bounded by the curve $x = f(t)$, $g(t)$, the y-axis and the abscissae at the points, where $t = a$, $t = b$ is

$$\int_a^b \pi y^2 \frac{dy}{dt}\,dt$$

Example 1. Find the volume of the solid of revolution obtained the area included between the curves $y^2 = x^3$ and $x^2 = y^3$ about the x-axis.

Solution. The equations of the given curves are

$$y^2 = x^3 \qquad \ldots(1)$$

and $$x^2 = y^3 \qquad \ldots(2)$$

Here the curve (1) is symmetrical about x-axis and the curve (2) is symmetrical about y-axis. Let us find the points of intersection of these curves by solving (1) and (2)

From (1) and (2), we get $x^2 = (x^{3/2})^3$

or $(x^2 - x^{9/2}) = 0$

or $x^2(1 - x^{5/2}) = 0 \quad \Rightarrow \quad x = 0, 1$

If $x = 0$, then $y = 0$

If $x = 1$, then $y = 1$.

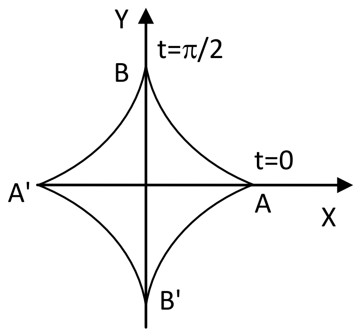
Thus the curves (1) and (2) meets in points (0, 0) and (1, 1)

$$\therefore \quad \text{Required volume} = \pi\int_0^1 (y_I^2 - y_{II}^2)\,dx$$

$$= \pi\int_0^1 (x^{4/3} - x^3)dx = \pi\left[\frac{3}{7}x^{7/3} - \frac{x^4}{4}\right]_0^1$$

$$= \pi\left(\frac{3}{7} - \frac{1}{4}\right) = \frac{5}{28}\pi.$$

Example 2. Find the volume of the spindle shaped solid generated by revolving the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ about the x-axis.



Solution. The given equation of the curve is $x^{2/3} + y^{2/3} = a^{2/3}$

Its parametric equations are

$$x = a\cos^3 t,\ \ y = a\sin^3 t \qquad \ldots(1)$$

The curve (1) is symmetrical about x-axis and y-axis. In the

Remarks

first quadrant, t varies from 0 to $\dfrac{\pi}{2}$.

$\therefore$ Required volume = 2 × volume generated by arc in the 1st quadrant

$$= 2\int_0^{\pi/2} \pi y^2 \frac{dx}{dt} dt$$

$$= 2\pi \int_0^{\pi/2} (a \sin^3 t)^2 (-3a \cos^2 t . \sin t) dt$$

$$= -6\pi a^3 \int_0^{\pi/2} \sin^7 t \cos^2 t \, dt$$

$$= (-6\pi a^3) . \frac{6.4.2}{9.7.5.3.1} = -\frac{32}{106} \pi a^3$$

$$= \frac{32}{105} \pi a^3 \text{ (numerically)} .$$

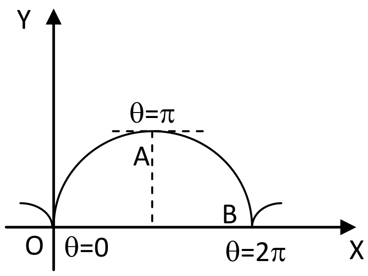
Example 3. Find the volume of solid formed by the revolution of one arch of the cycloid x = a(θ– sin θ), y = a(1 – cos θ) about its base.

Solution. The equations of the cycloid are x = a(θ– sin θ), y = a(1 – cos θ) …(1)

1. The cycloid is symmetrical about the line which is perpendicular to x-axis and passes through the point where θ = π.
2. If x = 0, (θ– sin θ) = 0 i.e., θ = 0 and then y = 0. Thus the curve passes through the origin.
3. For the first half of the cycloid in the first quadrant θ varies from 0 to π.
4. The base is x-axis.

$\therefore$ Required volume of the solid obtained by revolving about its base

$$= 2\int_0^{\pi} \pi y^2 \frac{dx}{d\theta} d\theta$$



$$= 2\pi \int_0^{\pi} a^2 (1 - \cos\theta)^2 a(1 - \cos\theta) d\theta$$

$$= 2\pi a^3 \int_0^{\pi} \left(2\sin^2 \frac{\theta}{2}\right)^2 \left(2 \sin^2 \frac{\theta}{2}\right) d\theta$$

Put $\dfrac{\theta}{2} = t$, $\therefore$ $d\theta = 2dt$

Now when θ = 0, t = 0 and when θ = π, t = $\dfrac{\pi}{2}$

$\therefore$ Required volume = $2\pi a^3 \displaystyle\int_0^{\pi/2} (2\sin^2 t)^3 . 2dt$

$$= 32\pi a^3 \int_0^{\pi/2} \sin^6 t \, dt$$

$$= 32\pi a^3 . \frac{5.3.1}{6.4.2} . \frac{\pi}{2} = 5\pi^2 a^3 .$$

Remarks

Exercise 9.3

1. The region bounded by the parabola and tangents at the extremities of tis latus rectum revolves about the axis of the parabola. Find the volume of the solid generated.

Ans. $\frac{2}{3}\pi a^3$.

2. Find the volume of the solid formed by revolution of one arch of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$ about its base.

Ans. $5\pi^2 a^3$.

3. Show that the volume of the solid formed by the revolution of one arch of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ about the tangent at the vertex is $\pi^2 a^3$.

4. Prove that the volume of the solid generated by the revolution about the x-axis of the loop of the curve $x = t^2$, $y = t - \frac{1}{3}t^3$ is $\frac{3\pi}{4}$.

5. Show that the volume of the solid generated by revolving about the y-axis the area between the first arch of the cycloid $x = \theta + \sin\theta$, $y = 1 - \cos\theta$ and the x-axis is $\pi\left[\frac{3}{2}\pi^2 - \frac{8}{3}\right]$.

9.7. VOLUME FORMULA FOR POLAR CURVES

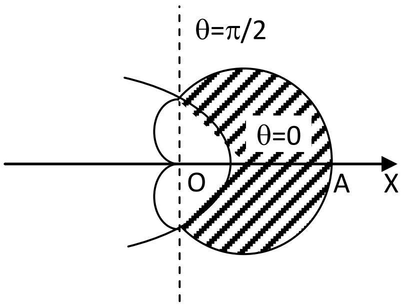
The volume of the solid generated by revolving the area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \alpha$, $\theta = \beta$

(i) about the line $\theta = 0$(i.e., initial line OX) is

$$\int_\alpha^\beta \frac{2}{3}\pi r^3 \sin\theta d\theta$$

(ii) about the line $\theta = \frac{\pi}{2}$(i.e., line OY) is

$$\int_\alpha^\beta \frac{2}{3}\pi r^3 \cos\theta d\theta$$



Example 1. Show that if the area lying within the cardioid $r = 2a(1 + \cos\theta)$ and without the parabola $r(1 + \cos\theta) = 2a$, revolves about the initial line, the volume generated is $18\pi a^3$.

Solution. The given equation of the cardioid is

$$r = 2a(1 + \cos\theta) \quad \ldots(1)$$

and that of the parabola is $$r = \frac{2a}{1+\cos\theta}$$

Here the curves (1) and (2) are both symmetrical about the initial line. The upper half of the region (shaded) revolves about the initial line and generates the required volume.

Also the curves (1) and (2) intersect when

$$2a(1 + \cos\theta) = \frac{2a}{1+\cos\theta}$$

i.e., $(1 + \cos\theta)^2 = 1$

i.e., $(1 + \cos\theta) = \pm 1$

If $(1 + \cos\theta) = 1$, then

Remarks

$$\cos\theta = 0 \qquad \Rightarrow \qquad \theta = \pm\frac{\pi}{2}$$

If $(1 + \cos\theta) = -1$, then

$$\cos\theta = -2 \text{ (not possible)}$$

Thus the curves (1) and (2) intersect at the points where $\theta = \frac{\pi}{2}$ and $\theta = -\frac{\pi}{2}$.

For the upper half, θ varies from 0 to $\frac{\pi}{2}$.

$\therefore$ Required volume $= \displaystyle\int_0^{\pi/2} \frac{2}{3}\pi \times [(\text{r for outer curve})^3 - (\text{r for inner curve})^3] \sin\theta d\theta$

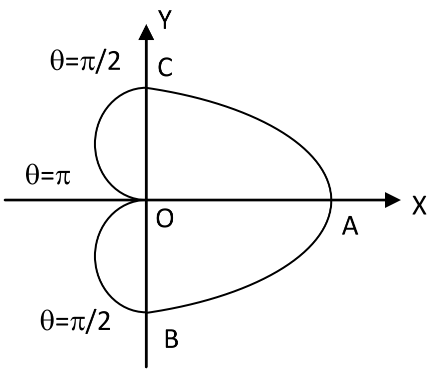
$$= \frac{2}{3}\pi \int_0^{\pi/2} \left[8a^3(1+\cos\theta)^3 - \frac{8a^3}{(1+\cos\theta)^3}\right] \sin\theta \, d\theta$$

$$= \frac{16\pi a^3}{3}\left[\frac{(1+\cos\theta)^4}{-4} - \frac{(1+\cos\theta)^{-2}}{(-2)(-1)}\right]_0^{\pi/2}$$

$$= \frac{16\pi a^3}{3}\left[-\frac{1}{4}(1-16) - \frac{1}{2}\left(1-\frac{1}{4}\right)\right]$$

$$= \frac{16\pi a^3}{3}\left[\frac{15}{4} - \frac{3}{8}\right] = \frac{16\pi a^3}{3} \times \frac{27}{8} = 18\pi a^3.$$

Example 2. The cardioid $r = a(1 + \cos\theta)$ revolves about the initial line. Find the volume of the solid generated.



Solution. The given equation of the cardioid is

$$r = a(1 + \cos\theta) \qquad \text{….(1)}$$

1. If θ is changed to $-\theta$, the equation of the curve remains unchanged. Thus the curve is symmetrical about the initial line.
2. If $\theta = \pi$, $r = 0$. Thus pole lies on the curve.
3. When θ increases from 0 to π, r remains positive and decreases from 2a to 0.

$\therefore$ Required volume $= \displaystyle\int_0^{\pi} \frac{2}{3}\pi r^3 \sin\theta d\theta$

$$= \frac{2}{3}\pi\int_0^{\pi} a^3(1+\cos\theta)^3.\sin\theta d\theta$$

$$= \frac{2}{3}\pi a^3 \left|-\frac{(1+\cos\theta)^4}{4}\right|_0^{\pi} = \frac{1}{6}\pi a^3 \,|\,(1+\cos\theta)^4\,|_0^{\pi}$$

$$= -\frac{1}{6}\pi a^3(0-16) = \frac{8}{3}\pi a^3.$$

Exercise 9.4

1. Show that volume of the slid formed by the revolution of the curve $r = a + b\cos\theta (a > b)$ about the initial line is $\frac{4}{3}\pi a(a^2 + b^2)$.

Remarks

2. Find the volume of the slid generated by the revolution of the cardioid $r = a(1 - \cos\theta)$ about the initial line.

Ans. $\frac{8}{3}\pi a^3$.

3. Find the volume of the solid generated by the revolution of $r = 2a\cos\theta$ about the initial line.

Ans. $\frac{4}{3}\pi a^3$.

4. The area of inner loop of the curve $r = 1 + 2\cos\theta$ is rotated through two right angles about the initial line. Show that the volume of the solid formed is $\frac{\pi}{12}$.

5. Find the volume of the solid generated by revolving one loop of the leminscate $r^2 = a^2\cos 2\theta$ about the line $\theta = \frac{\pi}{2}$.

Ans. $\frac{\pi^2 a^3}{4\sqrt{2}}$.

9.8. AREA OF A SURFACE OF REVOLUTION

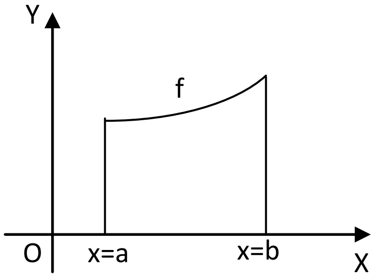
Definition. A surface of revolution is a surface that is formed by rotating a plane curve about a fixed line lying in its plane. The fixed line is called the axis of the surface of revolution and the curve is said to generate the surface.

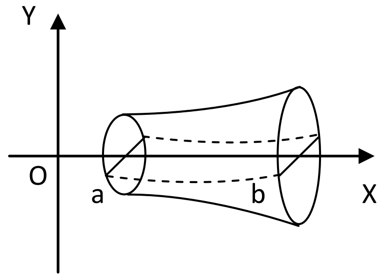
A right circular cylinder is an example of a surface of revolution for which the generating curve is a straight line parallel to the axis of revolution.

A sphere can be generated by rotating a semi-circle about its diameter.

Let f be a non–negative function such that its derivative f' is continuous on [a, b]. By revolving the graph of f from x = a to x = b about the x-axis, we obtain a surface of revolution. The area of such a surface is commonly defined to be

$$2\pi\int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx$$





9.8.1. Surface area of the solid of revolution of y = f(x) about x-axis from x = a to x = b is given by $\int_{x=a}^{x=b} 2\pi y\,ds$ **where s is the length of the arc of the curve measured from a fixed point on it to any point (x, y) and it being assumed that the arc does not cut x-axis.**

Three Forms of the Surface Formula

1. **For cartesian curves y = f(x)**, the formula for surface area of revolution about x-axis reduces to

$$S = \int_a^b 2\pi y\frac{ds}{dx}dx = \int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

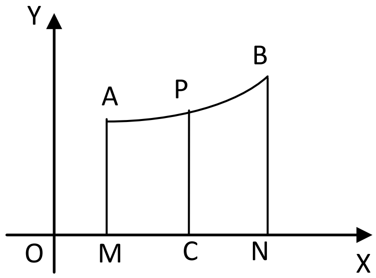
2. For parametric curves, the formula reduces to

$$S = \int_\alpha^\beta 2\pi y\frac{ds}{dt}dt = \int_\alpha^\beta 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

where t = α when x = a and $\theta = \beta$ when x = b.

Revolution About Any Axis

The curved surface of the solid generated by the revolution, about an axis MN of the area bounded by the curve AB, the axis MN and the perpendiculars AM, BN on the axis is



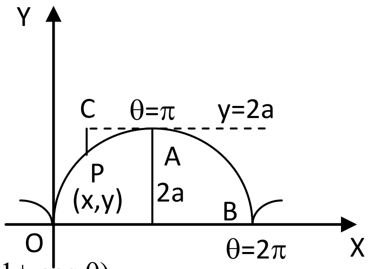
$$\int_{x=OM}^{x=ON} 2\pi(PC)\,ds$$

where PC is the perpendicular from any point P of the curve AB on MN and arc AB = s.

Example 1. Show that the area of the surface generated by the revolution of an arch of the cycloid x = a(θ– sin θ), y = a(1 – cos θ) about the line y = 2a is $\frac{32}{3}\pi a^2$

Solution. Proceeding as in example 3, we get

$$\frac{ds}{d\theta} = 2a\sin\frac{\theta}{2}$$



Also the arch OAB of the cycloid revolves about the line y = 2a(which is a tangent to the cycloid at the vertex A).

Let P(x, y) be any point on the arc OA

Draw PC perpendicular on AC so that

$$PC = 2a - y$$

$$= 2a - a(1 - \cos\theta) = a\,(1 + \cos\theta)$$

For the arc OA, θ varies from 0 to π

$\therefore$ Required surface = 2 × surface generated by the revolution of the arc OA about the line y = 2a

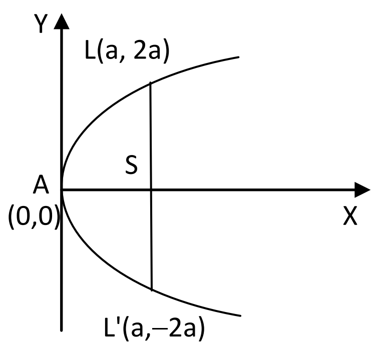
$$= 2\int_0^{\pi} 2\pi(PC)\frac{ds}{d\theta}d\theta = 4\pi\int_0^{\pi} a(1+\cos\theta).2a\sin\frac{\theta}{2}d\theta$$

$$= 4\pi\int_0^{\pi} a\left(2\cos^2\frac{\theta}{2}\right).2a\sin\frac{\theta}{2}d\theta = 16\pi a^2\int_0^{\pi}\cos^2\frac{\theta}{2}\sin\frac{\theta}{2}d\theta$$

$$= -32\pi a^2\left|\frac{\cos^2\frac{\theta}{2}}{3}\right|_0^{\pi} = -\frac{32}{3}\pi a^2(0-1)$$

$$= \frac{32}{3}\pi a^2.$$

Example 2. Find the area of the surface formed by the revolution of $y^2 = 4ax$, about the x-axis by the arc from the vertex to one end of latus rectum.



Solution. The given equation of the parabola is

$$y^2 = 4ax \qquad \ldots(1)$$

From (1), we get

$$\frac{dy}{dx} = \frac{2a}{y}$$

Remarks

Also $$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{4a^2}{y^2}}$$

$$= \sqrt{1+\frac{4a^2}{4ax}} = \sqrt{\frac{x+a}{x}}$$

Now for the arc from the vertex to one end, say L of the latus rectum x varies from 0 to a.

∴ Required surface $$= \int_0^a 2\pi y \frac{ds}{dx} dx = 2\pi\int_0^a \sqrt{4ax}\sqrt{\frac{x+a}{x}}dx$$

$$= 4\pi\sqrt{a}\int_0^a (x+a)^{1/2} dx = 4\pi\sqrt{a}\left[\frac{(x+a)^{3/2}}{\frac{3}{2}}\right]_0^a$$

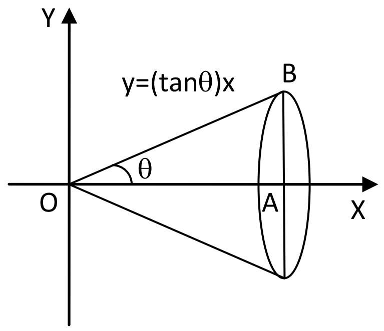
$$= \frac{8\pi\sqrt{a}}{3}[(2a)^{3/2} - a^{3/2}] = \frac{8}{3}\pi a^2[2\sqrt{2}-1].$$

Example 3. Find the surface of the right circular cone formed by the revolution of a right angled triangle about a side which contains the right angle.

Solution. Consider a right angled ΔOAB, right angled at A, formed by the lines y = (tan θ)x, and line OA lying on x-axis such that OA = h represents the height of the cone thus generated.

Let $r = AB$ (radius of the base)

$l = OB$ (slant height)



Then $\sec\theta = \frac{l}{h}$

Also $y = x\tan\theta$

$$\Rightarrow \quad \frac{dy}{dx} = \tan\theta$$

$$\therefore \quad \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$= \sqrt{1+\tan^2\theta} = \sec\theta$$

∴ Required surface area $$= \int_0^h 2\pi y\frac{ds}{dx}.dx$$

$$= 2\pi\int_0^h x\tan\theta.\sec\theta dx = 2\pi\tan\theta\sec\theta\int_0^h x\,dx$$

$$= 2\pi\tan\theta\sec\theta\left[\frac{x^2}{2}\right]_0^h = 2\pi\tan\theta\sec\theta\left[\frac{h^2}{2}\right]$$

$$= \pi h^2\tan\theta\sec\theta = \pi h^2.\frac{r}{h}.\frac{l}{h} = \pi rl.$$

Example 4. Prove that surface generated by the revolution of the tractrix $x = a\cos t + \frac{a}{2}\log\tan^2\frac{t}{2}$, $y = a\sin t$ about its asymptote is equal to the surface of a sphere of radius a.

Solution. The equations of the curve are

Remarks

$$x = a\cos t + \frac{a}{2}\log\tan^2\frac{t}{2} = a\cos t + a\log\tan\frac{t}{2} \qquad \ldots(1)$$

and $\qquad y = a\sin t$

Curve (1) is symmetrical about both the axes and its asymptote is $y = 0$ i.e., x-axis.

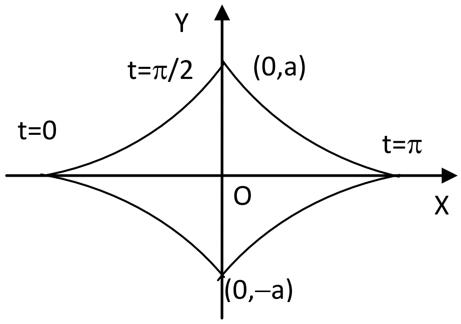
Differentiating (1) w.r.t. t, we get

$$\frac{dx}{dt} = -a\sin t + a \times \frac{1}{\tan\frac{t}{2}}\sec^2\frac{t}{2}.\frac{1}{2}$$

$$= -a\sin t + \frac{a}{2\sin\frac{t}{2}\cos\frac{t}{2}}$$

$$= -a\sin t + \frac{a}{\sin t}$$

$$= \frac{a(1-\sin^2 t)}{\sin t} = \frac{a\cos^2 t}{\sin t}$$



and $\qquad \dfrac{dy}{dt} = a\cos t$

Now $\qquad \dfrac{ds}{dt} = \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2} = \sqrt{\dfrac{a^2\cos^4 t}{\sin^2 t} + a^2\cos^2 t}$

$$= \sqrt{\frac{a^2\cos^2 t(\cos^2 t + \sin^2 t)}{\sin^2 t}} = \frac{a\cos t}{\sin t}$$

Also for the curve in the second quadrant, t varies from 0 to $\frac{\pi}{2}$.

$\therefore \qquad$ Required surface $= 2\displaystyle\int_0^{\pi/2} 2\pi y\frac{ds}{dt}dt$

$$= 4\pi\int_0^{\pi/2} a\sin t.\frac{a\cos t}{\sin t}dt$$

$$= 4\pi a^2\int_0^{\pi/2}\cos t\,dt = 4\pi a^2\,|\sin t|_0^{\pi/2}$$

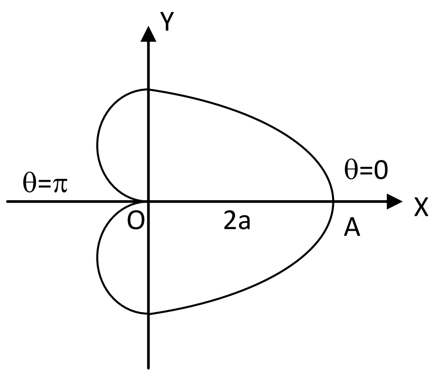
$$= 4\pi a^2(1-0) = 4\pi a^2.$$

Example 5. Show that surface area of the solid of revolution of $r = a(1 + \cos\theta)$ about the initial line is $\frac{32}{5}\pi a^2$.

Solution. The given equation of the cardioid is

$$r = a(1 + \cos\theta) \qquad \ldots(1)$$

The curve is symmetrical about the initial line and the required surface area is obtained by the revolving the curve from $\theta = 0$ to $\theta = \pi$.



From (1), $\dfrac{dr}{d\theta} = -a\sin\theta$

and $\qquad \dfrac{ds}{d\theta} = \sqrt{r^2 + \left(\dfrac{dr}{d\theta}\right)^2}$

Remarks

$$= \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta} = a\sqrt{1+\cos^2\theta+2\cos\theta+\sin^2\theta}$$

$$= a\sqrt{2(1+\cos\theta)} = 2a\cos\frac{\theta}{2}$$

$\therefore$ Required surface

$$= \int_0^{\pi} 2\pi(r\sin\theta)\frac{ds}{d\theta}d\theta$$

$$= 2\pi\int_0^{\pi}(r\sin\theta)\left(2a\cos\frac{\theta}{2}\right)d\theta$$

$$= 4\pi a^2\int_0^{\pi}(1+\cos\theta)\sin\theta\cos\frac{\theta}{2}d\theta$$

$$= 4\pi a^2\int_0^{\pi}2\cos^2\frac{\theta}{2}.\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)\cos\frac{\theta}{2}d\theta$$

$$= 16\pi a^2\int_0^{\pi}\cos^4\frac{\theta}{2}\sin\frac{\theta}{2}d\theta$$

$$= 16\pi a^2\left|\frac{\cos^5\frac{\theta}{2}}{5.\frac{1}{2}}\right|_0^{\pi} = -\frac{32\pi a^2}{5}(0-1) = \frac{32}{5}\pi a^2 .$$

Exercise 9.5

1. Find the surface area of the solid generated by revolving one arc of the curve $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$ about x-axis.

Ans. $\frac{64}{3}\pi a^2$.

2. Find the surface area of the solid generated by revolution of the leminiscate $r^2 = a^2\cos 2\theta$ about the initial line.

Ans. $4\pi a^2\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)$.

3. Find the surface of the solid generated by the revolution of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ about the x-axis.

Ans. $\frac{12}{5}\pi a^2$.

4. Find the area of the curved surface generated by revolution of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1-\cos\theta)$ about its base.

Ans. $\frac{32}{3}\pi a^2$.

5. Find the surface generated by the revolution of an arc of the catenary $y = c\cosh\frac{x}{c}$ about the x-axis.

Ans. $c\pi\left(x + \frac{c}{2}\sinh\frac{2x}{c}\right)$.

6. Prove that the surface of the solid obtained by revolving the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ about x-axis is $2\pi ab\left[\sqrt{1-e^2} + \frac{1}{e}\sin^{-1}e\right]$ where e is the eccentricity of the ellipse.

Remarks

7. Find the area of the surface formed by revolution of $y^2 = 4ax$ about y-axis by the arc from the vertex to $x = \frac{a}{4}$.

Ans. $\frac{\pi a^2}{4}\left[\frac{3\sqrt{5}}{4} - \log\frac{3+\sqrt{5}}{2}\right]$.

8. Find the surface of a sphere of radius a. OR The circle $x^2 + y^2 = a^2$ is revolved about x-axis. Find the area of the sphere generated.

Ans. $4\pi a^2$.

Keywords : Volume, surface area, axis of revolution, parametric, polar.

Summary : Here volume obtained by revolving about x-axis, y-axis and about any axis of rotation.

$$V = \int_a^b \pi y^2 dx \qquad \text{(revolving about x-axis)}$$

$$V = \int_a^b \pi x^2 dy \qquad \text{(revolving about y-axis)}$$

$$V = \int_{OM}^{ON} \pi (PC)^2 d(OC)$$

Also surface area of the solid of revolution of y = f(x) about x-axis, y-axis and any axis.

$$S.A = \int_a^b 2\pi y dS \qquad \text{(revolution about x-axis)}$$

$$S.A = \int_c^d 2\pi x dS \qquad \text{(revolution asbout y-axis)}$$

$$S.A = \int_{OM}^{ON} 2\pi (PC) dS \qquad \text{(about any axis of revolution)}$$